

# Accelerated FRW Solutions in Chern-Simons Gravity

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## Abstract

We consider a five-dimensional Einstein-Chern-Simons action which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Chern-Simons gravity action instead of the Einstein-Hilbert action and where the matter sector is given by the so called perfect fluid. It is shown that (i) the Einstein-Chern-Simons (EChS) field equations subject to suitable conditions can be written in a similar way to the Einstein-Maxwell field equations; (ii) these equations have solutions that describe accelerated expansion for the three possible cosmological models of the universe, namely, spherical expansion, flat expansion and hyperbolic expansion when  $\alpha$ , a parameter of theory, is greater than zero. This result allow us to conjecture that this solutions are compatible with the era of Dark Energy and that the energy-momentum tensor for the field  $h^a$ , a bosonic gauge field from the Chern-Simons gravity action, corresponds to a form of positive cosmological constant.

It is also shown that the EChS field equations have solutions compatible with the era of matter: (i) In the case of an open universe, the solutions correspond to an accelerated expansion ( $\alpha > 0$ ) with a minimum scale factor at initial time that, when the time goes to infinity, the scale factor behaves as a hyperbolic sine function. (ii) In the case of a flat universe, the solutions describing an accelerated expansion whose scale factor behaves as a exponential function when time grows. (iii) In the case of a closed universe it is found only one solution for a universe in expansion, which behaves as a hyperbolic cosine function when time grows.

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## I. INTRODUCTION

Some time ago was shown that the standard, five-dimensional General Relativity can be obtained from Chern-Simons gravity theory for a certain Lie algebra  $\mathfrak{B}$  [1], whose generators  $\{\mathbf{J}_{ab}, \mathbf{P}_a, \mathbf{Z}_{ab}, \mathbf{Z}_a\}$  satisfy the commutation relationships

$$\begin{aligned} [\mathbf{J}_{ab}, \mathbf{J}_{cd}] &= \eta_{ad}\mathbf{J}_{bc} - \eta_{ac}\mathbf{J}_{bd} + \eta_{bc}\mathbf{J}_{ad} - \eta_{bd}\mathbf{J}_{ac}, \\ [\mathbf{P}_a, \mathbf{J}_{bc}] &= \eta_{ab}\mathbf{P}_c - \eta_{ac}\mathbf{P}_b, \\ [\mathbf{J}_{ab}, \mathbf{Z}_{cd}] &= \eta_{ad}\mathbf{Z}_{bc} - \eta_{ac}\mathbf{Z}_{bd} + \eta_{bc}\mathbf{Z}_{ad} - \eta_{bd}\mathbf{Z}_{ac}, \\ [\mathbf{Z}_a, \mathbf{J}_{bc}] &= \eta_{ab}\mathbf{Z}_c - \eta_{ac}\mathbf{Z}_b, \\ [\mathbf{P}_a, \mathbf{P}_b] &= \mathbf{Z}_{ab}, \\ [\mathbf{P}_a, \mathbf{Z}_{bc}] &= \eta_{ab}\mathbf{Z}_c - \eta_{ac}\mathbf{Z}_b. \end{aligned}$$

This algebra was obtained from the anti de Sitter (AdS) algebra and a particular semi-group  $S$  by means of the S-expansion procedure introduced in Refs. [2], [3].

In order to write down a Chern-Simons lagrangian for the  $\mathfrak{B}$  algebra, we start from the one-form gauge connection

$$\mathbf{A} = \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + \frac{1}{l}e^a\mathbf{P}_a + \frac{1}{2}k^{ab}\mathbf{Z}_{ab} + \frac{1}{l}h^a\mathbf{Z}_a, \quad (1)$$

and the two-form curvature

$$\begin{aligned} \mathbf{F} &= \frac{1}{2}R^{ab}\mathbf{J}_{ab} + \frac{1}{l}T^a\mathbf{P}_a + \frac{1}{2}\left(D_\omega k^{ab} + \frac{1}{l^2}e^ae^b\right)\mathbf{Z}_{ab} \\ &\quad + \frac{1}{l}\left(D_\omega h^a + k^a{}_b e^b\right)\mathbf{Z}_a. \end{aligned} \quad (2)$$

Consistency with the dual procedure of S-expansion in terms of the Maurer-Cartan forms [3] demands that  $h^a$  inherits units of length from the *fünfbein*; that is why it is necessary to introduce the  $l$  parameter again, this time associated with  $h^a$ .

It is interesting to observe that  $\mathbf{J}_{ab}$  are still Lorentz generators, but  $\mathbf{P}_a$  are no longer AdS boosts; in fact,  $[\mathbf{P}_a, \mathbf{P}_b] = \mathbf{Z}_{ab}$ . However,  $e^a$  still transforms as a vector under Lorentz transformations, as it must be in order to recover gravity in this scheme.

A Chern-Simons lagrangian in  $d = 5$  dimensions is defined to be the following local function of a one-form gauge connection  $\mathbf{A}$ :

$$L_{\text{ChS}}^{(5)}(\mathbf{A}) = k \left\langle \mathbf{A}\mathbf{F}^2 - \frac{1}{2}\mathbf{A}^3\mathbf{F} + \frac{1}{10}\mathbf{A}^5 \right\rangle, \quad (3)$$

where  $\langle \cdots \rangle$  denotes a invariant tensor for the corresponding Lie algebra,  $\mathbf{F} = d\mathbf{A} + \mathbf{A}\mathbf{A}$  is the corresponding the two-form curvature and  $k$  is a constant [4].

Using theorem VII.2 of Ref. [2], it is possible to show that the only non-vanishing components of a invariant tensor for the  $\mathfrak{B}$  algebra are given by

$$\begin{aligned}\langle \mathbf{J}_{a_1 a_2} \mathbf{J}_{a_3 a_4} \mathbf{P}_{a_5} \rangle &= \alpha_1 \frac{4l^3}{3} \epsilon_{a_1 \cdots a_5}, \\ \langle \mathbf{J}_{a_1 a_2} \mathbf{J}_{a_3 a_4} \mathbf{Z}_{a_5} \rangle &= \alpha_3 \frac{4l^3}{3} \epsilon_{a_1 \cdots a_5}, \\ \langle \mathbf{J}_{a_1 a_2} \mathbf{Z}_{a_3 a_4} \mathbf{P}_{a_5} \rangle &= \alpha_3 \frac{4l^3}{3} \epsilon_{a_1 \cdots a_5},\end{aligned}\tag{4}$$

where  $\alpha_1$  and  $\alpha_3$  are arbitrary independent constants of dimensions  $[length]^{-3}$ .

Using the extended Cartan's homotopy formula as in Ref. [5], and integrating by parts, it is possible to write down the Chern-Simons Lagrangian in five dimensions for the  $\mathcal{B}$  algebra as

$$\begin{aligned}L_{\text{EChS}}^{(5)} &= \alpha_1 l^2 \epsilon_{abcde} e^a R^{bc} R^{de} \\ &\quad + \alpha_3 \epsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right) \\ &\quad + dB_{\text{EChS}}^{(4)}\end{aligned}\tag{5}$$

where the surface term  $B_{\text{EChS}}^{(4)}$  is given by

$$\begin{aligned}B_{\text{EChS}}^{(4)} &= \alpha_1 l^2 \epsilon_{abcde} e^a \omega^{bc} \left( \frac{2}{3} d\omega^{de} + \frac{1}{2} \omega_f^d \omega^{fe} \right) \\ &\quad + \alpha_3 \epsilon_{abcde} \left[ l^2 (h^a \omega^{bc} + k^{ab} e^c) \left( \frac{2}{3} d\omega^{de} + \frac{1}{2} \omega_f^d \omega^{fe} \right) \right. \\ &\quad \left. + l^2 k^{ab} \omega^{cd} \left( \frac{2}{3} de^e + \frac{1}{2} \omega_f^d e^e \right) + \frac{1}{6} e^a e^b e^c \omega^{de} \right]\end{aligned}\tag{6}$$

and where  $\alpha_1, \alpha_3$  are parameters of the theory,  $l$  is a coupling constant,  $R^{ab} = d\omega^{ab} + \omega_c^a \omega^{cb}$  corresponds to the curvature 2-form in the first-order formalism related to the 1-form spin connection [4], [6], [7], and  $e^a, h^a$  and  $k^{ab}$  are others gauge fields presents in the theory [1].

From (5) we can see that the third term is a surface term and can be removed from this Lagrangian. So that,

$$\begin{aligned}L_{\text{EChS}}^{(5)} &= \alpha_1 l^2 \epsilon_{abcde} R^{ab} R^{cd} e^e \\ &\quad + \alpha_3 \epsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right)\end{aligned}\tag{7}$$

is the Einstein-Chern-Simons Lagrangian studied in Ref [1].

It should be noted the absence of kinetic terms for the fields  $h^a$  and  $k^{ab}$  in equation (7). The term kinetic for the  $h^a$  and  $k^{ab}$  fields are present in the surface term of the Lagrangian (5) given by (6).

The Lagrangian (7) show that standard, five-dimensional General Relativity emerges as the  $l \rightarrow 0$  limit of a CS theory for the generalized Poincaré algebra  $\mathfrak{B}$ . Here  $l$  is a length scale, a coupling constant that characterizes different regimes within the theory. The  $\mathfrak{B}$  algebra, on the other hand, is constructed from the AdS algebra and a particular semigroup  $S$  by means of the S-expansion procedure. The field content induced by the  $\mathfrak{B}$  algebra includes the vielbein  $e^a$ , the spin connection  $\omega^{ab}$  and two extra bosonic fields  $h^a$  and  $k^{ab}$ , which can be interpreted as boson fields coupled to the field curvature and the parameter  $l^2$  can be interpreted as a kind of coupling constant.

Recently was found [8] that the standard five-dimensional FRW equations and some of their solutions can be obtained, in a certain limit, from the so-called Chern-Simons-FRW field equations, which are the cosmological field equations corresponding to a Chern-Simons gravity theory.

It is the purpose of this paper to show that the Einstein-Chern-Simons (EChS) field equations, subject to (i) the torsion-free condition ( $T^a = 0$ ) and (ii) the variation of the matter Lagrangian with respect to (*w.r.t.*) the spin connection is zero ( $\delta L_M / \delta \omega^{ab} = 0$ ) can be written in a similar way to the Einstein-Maxwell field equations. The interpretation of the  $h^a$  field as a perfect fluid allow us to show that the Einstein-Chern-Simons field equations have an universe in accelerated expansion as a of their solutions.

This paper is organized as follows: In Section II we briefly review the Einstein-Chern-Simons field equations. In Section III we study the Einstein-Chern-Simons field equations in the range of validity of general relativity. In Section V we consider accelerated solutions for Einstein-Chern-Simons field equations. We try to find solutions that describes accelerated expansion for cases of open universes, flat universes and closed universes. In Section VI we consider the consistency of the solutions with the "Era of Matter". A summary and an appendix conclude this work.

## II. EINSTEIN-CHERN-SIMONS FIELD EQUATIONS

In Ref. [8] was found that in the presence of matter the lagrangian is given by

$$L = L_{\text{ChS}}^{(5)} + \kappa L_M \quad (8)$$

where  $L_{\text{ChS}}^{(5)}$  is the five-dimensional Chern-Simons lagrangian given by (7),  $L_M = L_M(e^a, h^a, \omega^{ab})$  is the matter Lagrangian and  $\kappa$  is a coupling constant related to the effective Newton's constant. The variation of the lagrangian (8) w.r.t. the dynamical fields vielbein  $e^a$ , spin connection  $\omega^{ab}$ ,  $h^a$  and  $k^{ab}$ , leads to the following field equations

$$\varepsilon_{abcde} \left( 2\alpha_3 R^{ab} e^c e^d + \alpha_1 l^2 R^{ab} R^{cd} + 2\alpha_3 l^2 D_\omega k^{ab} R^{cd} \right) = \kappa \frac{\delta L_M}{\delta e^e}, \quad (9)$$

$$\alpha_3 l^2 \varepsilon_{abcde} R^{ab} R^{cd} = \kappa \frac{\delta L_M}{\delta h^e}, \quad (10)$$

$$2\alpha_3 l^2 \varepsilon_{abcde} R^{cd} T^e = \kappa \frac{\delta L_M}{\delta k^{ab}}, \quad (11)$$

$$\begin{aligned} 2\varepsilon_{abcde} \left( \alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 D_\omega k^{ab} T^e + \alpha_3 e^c e^d T^e + \alpha_3 l^2 R^{cd} D_\omega h^e \right) \\ + 2\alpha_3 \varepsilon_{abcde} l^2 R^{cd} k_f^e e^f = \kappa \frac{\delta L_M}{\delta \omega^{ab}}. \end{aligned} \quad (12)$$

For simplicity, we will assume that the torsion vanishes ( $T^a = 0$ ) and  $k^{ab} = 0$ . In this case the Eqs.(9 - 12) takes the form

$$\varepsilon_{abcde} \left( 2\alpha_3 R^{ab} e^c e^d + \alpha_1 l^2 R^{ab} R^{cd} \right) = \kappa \frac{\delta L_M}{\delta e^e}, \quad (13)$$

$$\alpha_3 l^2 \varepsilon_{abcde} R^{ab} R^{cd} = \kappa \frac{\delta L_M}{\delta h^e}, \quad (14)$$

$$\frac{\delta L_M}{\delta k^{ab}} = 0 \quad (15)$$

$$2\alpha_3 l^2 \varepsilon_{abcde} R^{cd} D_\omega h^e = \kappa \frac{\delta L_M}{\delta \omega^{ab}}. \quad (16)$$

This field equations system can be written in the form

$$\varepsilon_{abcde} R^{ab} e^c e^d = 4\kappa_5 \left( \frac{\delta L_M}{\delta e^e} + \alpha \frac{\delta L_M}{\delta h^e} \right), \quad (17)$$

$$l^2 \varepsilon_{abcde} R^{ab} R^{cd} = 8\kappa_5 \frac{\delta L_M}{\delta h^e}, \quad (18)$$

$$l^2 \varepsilon_{abcde} R^{cd} D_\omega h^e = 4\kappa_5 \frac{\delta L_M}{\delta \omega^{ab}} \quad (19)$$

where we introduce  $\kappa_5 = \kappa/8\alpha_3$  and  $\alpha = -\alpha_1/\alpha_3$ .

The field equation (9) contains three terms. The first one, proportional to the Einstein tensor. The second one corresponds to a quadratic term in the curvature, and a third one, a term that describes the dynamics of the field  $k^{ab}$ . Since we assume  $k^{ab} = 0$  the last term in left side of Eq. (9) vanishes.

In order to write this field equation manner analogous to Einstein's equations, one chooses to leave the term proportional to the Einstein tensor on the left side of Eq. (9)

$$\epsilon_{abcde} R^{bc} e^d e^e = \frac{\kappa}{2\alpha_3} \frac{\delta L_M}{\delta e^a} - \frac{\alpha_1}{2\alpha_3} l^2 \epsilon_{abcde} R^{bc} R^{de}$$

and using the Eq. (14) we obtain Eq. (17).

This result allows us to interpret  $\delta L_M/\delta h^a$  as the energy momentum tensor for a second type of matter, not ordinary. Henceforth we will say that  $\delta L_M/\delta h^a$  corresponds to the energy-momentum tensor for the field  $h^a$ .

The equation of motion for the  $h^a$ -field is given by Eq.(19). The condition  $\delta L_M/\delta \omega^{ab} = 0$  (usual in gravity theories), imposed for consistency with the condition  $T^a = 0$ , leads to the equation of motion (22) for the  $h^a$ -field. This means that  $h^a$ -field is governed by the following field equations

$$\varepsilon_{abcde} R^{ab} e^c e^d = 4\kappa_5 \left( \frac{\delta L_M}{\delta e^e} + \alpha \frac{\delta L_M}{\delta h^e} \right), \quad (20)$$

$$\frac{l^2}{8\kappa_5} \varepsilon_{abcde} R^{ab} R^{cd} = \frac{\delta L_M}{\delta h^e}, \quad (21)$$

$$\varepsilon_{abcde} R^{cd} D_\omega h^e = 0. \quad (22)$$

This means that the Einstein-Chern-Simons field equations, subject to the conditions  $T^a = 0$ ,  $k^{ab} = 0$  and  $\delta L_M/\delta \omega^{ab} = 0$ , can be re-written in a way similar to the Einstein-Maxwell field equations.

From (20-22) we can see that if  $L_M = 0$ , then in five dimensions there is no solution of Schwarzschild type [1], [9].

### III. EINSTEIN-CHERN-SIMONS EQUATIONS IN THE RANGE OF VALIDITY OF GENERAL RELATIVITY

From (20-21) we can see that general relativity is valid when (i) the curvature  $R^{ab}$  takes values not excessively large (ii) the parameter  $l$  takes small values ( $l \rightarrow 0$ ) [1]; (iii) the constant  $\alpha$  takes values not excessively large. In fact, in this case we have that (21) takes the form

$$\frac{\delta L_M}{\delta h^e} \approx 0. \quad (23)$$

Introducing (23) into (20) we obtain the Einstein's field equation

$$\varepsilon_{abcde} R^{ab} e^c e^d \approx 4\kappa_5 \frac{\delta L_M}{\delta e^e}. \quad (24)$$

If  $R^{ab}$  is not large then  $\delta L_M / \delta e^a$  is also not large. This means that General Relativity can be seen as a low energy limit of Einstein-Chern-Simons gravity. So that, in the range of validity of the General Relativity, the equations (20-22) are given by

$$\varepsilon_{abcde} R^{ab} e^c e^d = 4\kappa_5 \frac{\delta L_M}{\delta e^e}, \quad (25)$$

$$\varepsilon_{abcde} R^{cd} D_\omega h^e = 0. \quad (26)$$

On the another hand, if  $R^{ab}$  is large enough, so that when it is multiplied by  $l^2$  (which is very small) will have a non-negligible results, then we will find that  $\delta L_M / \delta h^a$  is not negligible. This means that, in this case, we must consider the entire system of equations (20-22).

### IV. EINSTEIN-CHERN-SIMONS FIELD EQUATIONS FOR A FRIEDMANN-ROBERTSON-WALKER-LIKE SPACETIME

The shape of the field  $e^a$  is obtained from of the application of the cosmological principle to the metric tensor of spacetime: it is considered a splitting of the 5D-manifold in a maximally symmetric four-dimensional manifold and one temporal dimension ( $M = R \times \Sigma_4$ ). This leads



to five dimensional Friedmann-Robertson-Walker (FRW) metric. So that, the vielbein can be chosen like in [8]:

$$\begin{aligned}
e^0 &= dt, \\
e^1 &= \frac{a(t)}{\sqrt{1-kr^2}} dr, \\
e^2 &= a(t)r d\theta_2, \\
e^3 &= a(t)r \sin \theta_2 d\theta_3, \\
e^4 &= a(t)r \sin \theta_2 \sin \theta_3 d\theta_4
\end{aligned} \tag{27}$$

where  $a(t)$  is the scale factor of the universe and  $k$  is the sign of the curvature of space ( $\Sigma_4$ ): (i) +1 for a closed space ( $S^4$ ), (ii) 0 for a flat space ( $E^4$ ) and (iii) -1 for an open space (hyperbolic).

The application of the cosmological principle to the metric tensor of the spacetime also constrains the shape of the field  $h^a$  (see for example [8]). A detailed discussion can be also found in Ref. [10]. The bosonic field  $h^a$  is given by

$$\begin{aligned}
h^0 &= h(0) dt = h(0)e^0, \\
h^1 &= h(t) \frac{a(t)}{\sqrt{1-kr^2}} dr = h(t)e^1, \\
h^2 &= h(t)a(t)r d\theta_2 = h(t)e^2, \\
h^3 &= h(t)a(t)r \sin \theta_2 d\theta_3 = h(t)e^3, \\
h^4 &= h(t)a(t)r \sin \theta_2 \sin \theta_3 d\theta_4 = h(t)e^4
\end{aligned} \tag{28}$$

where  $h(0)$  is a constant and  $h(t)$  is a function of time  $t$  that must be determined. Substituting (28) into Eq. (22) we obtain the explicit form of the equations of motion for the  $h^a$ -field, which will be displayed in Eq.(39).

In accordance with the equation (20), we will consider a fluid composed of two perfect fluids, the first one related to ordinary energy-momentum tensor ( $T_{\mu\nu} \sim \frac{\delta L_M}{\delta e^a}$ ) and the second one related to field  $h^a$  ( $T_{\mu\nu}^{(h)} \sim \frac{\delta L_M}{\delta h^a}$ ). The energy-momentum tensors in the comoving frame, are given by

$$T_{\mu\nu} = \text{diag}(\rho, p, p, p, p), \tag{29}$$

$$T_{\mu\nu}^{(h)} = \text{diag}\left(\rho^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}\right), \tag{30}$$

where  $\rho$  is the matter density and  $p$  is the pressure of fluid. Then, the energy-momentum tensor for the composed fluid is

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \alpha T_{\mu\nu}^{(h)} \quad (31)$$

$$= \text{diag}\left(\rho + \alpha\rho^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}\right) \quad (32)$$

$$= \text{diag}(\tilde{\rho}, \tilde{p}, \tilde{p}, \tilde{p}, \tilde{p}). \quad (33)$$

In the torsion-free case, the energy momentum tensor of ordinary matter satisfies a conservation equation and the Einstein tensor has also zero divergence. In this case the energy momentum tensor for the non-ordinary matter must also satisfy a conservation equation. In fact, from Eq. (20) we find

$$\nabla_\mu T^\mu_\nu = 0 \quad , \quad \nabla_\mu T^{(h)\mu}_\nu = 0 \quad (34)$$

Introducing (27 - 33) into eqs. (20 - 22) we find the following field equations (see Ref. [8] and Appendix A)

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 \tilde{\rho}, \quad (35)$$

$$3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \tilde{p}, \quad (36)$$

$$\frac{3l^2}{\kappa_5} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \rho^{(h)}, \quad (37)$$

$$\frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)}, \quad (38)$$

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (h - h(0)) \frac{\dot{a}}{a} + \dot{h} \right] = 0. \quad (39)$$

We should note that equation (35) was studied in Ref. [11] in the context of inflationary cosmology .

The Equations (35) and (36) are very similar to the Friedmann equations in five dimensions. However now  $\rho$  and  $p$  are subject to restrictions imposed by the remaining equations.

## V. ACCELERATED SOLUTION FOR EINSTEIN-CHERN-SIMONS FIELD EQUATIONS

In order to recover the known results of the standard cosmology in the context of accelerated expansion we use the approach

$$T_{\mu\nu} \ll \alpha T_{\mu\nu}^{(h)}$$

This approach is analogous to the case when, in the era of Dark Energy, the energy momentum tensor is neglected compared to the cosmological constant. This means that the contribution from the ordinary matter is negligible compared to the contribution from the field  $h^a$ . In this case, the energy-momentum tensor  $\tilde{T}_{\mu\nu}$  fluid is given by

$$\begin{aligned} \tilde{T}_{\mu\nu} &= \text{diag}(\tilde{\rho}, \tilde{p}, \tilde{p}, \tilde{p}, \tilde{p}) \\ &= \alpha T_{\mu\nu}^{(h)} \\ &= \text{diag}(\alpha \rho^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}) \end{aligned} \quad (40)$$

and the equations (35 - 39) take the form

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 \alpha \rho^{(h)}, \quad (41)$$

$$3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \alpha p^{(h)}, \quad (42)$$

$$\frac{3l^2}{\kappa_5} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \rho^{(h)}, \quad (43)$$

$$\frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)}, \quad (44)$$

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (h - h(0)) \frac{\dot{a}}{a} + \dot{h} \right] = 0. \quad (45)$$

### A. Case $T_{\mu\nu} = 0$ and $k = -1$

Introducing (43) into (41) we obtain

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = 3l^2 \alpha \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 \quad (46)$$

which can be rewritten

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left( \frac{2}{\alpha l^2} - \frac{\dot{a}^2 + k}{a^2} \right) = 0. \quad (47)$$

### 1. Solution $\ddot{a} = 0$

Consider the solution  $\ddot{a} = 0$ , i.e., a solution without accelerated expansion. For the first term in left side of (47) we have

$$\frac{\dot{a}^2 + k}{a^2} = 0, \quad (48)$$

remembering  $k = -1$ , we have

$$\dot{a} = \sqrt{-k}. \quad (49)$$

The solution is

$$a(t) = \sqrt{-k} (t - t_0) + a_0. \quad (50)$$

In this case  $a(t)$  is increase linearly, i.e., there is no accelerated expansion.

Replacing this solutions into equations (41 - 44) we find

$$\rho^{(h)} = p^{(h)} = 0 \quad (51)$$

and equation (45) is satisfied for  $h(t)$  arbitrary.

### 2. Solution $\ddot{a} \neq 0$

From (47) we obtain we obtain

$$\dot{a}^2 - \frac{2}{\alpha l^2} a^2 = -k. \quad (52)$$

From (52) we can see two options (i)  $\alpha > 0$  and (ii)  $\alpha < 0$ .

#### a. Case $\alpha > 0$ :

Consider the case where the constant  $\alpha$  is positive. Using the following *ansatz*<sup>1</sup>

$$a(t) = A \sinh \left( \sqrt{\frac{2}{\alpha l^2}} (t - t') \right) \quad (53)$$

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<sup>1</sup> This *ansatz* can be obtained from

$$\dot{a} = \sqrt{\frac{2}{\alpha l^2} a^2 - k}$$

whose solution is ( $\alpha > 0$ ,  $k = -1$ )

$$\int_{t'}^t \frac{da}{\sqrt{\frac{2}{\alpha l^2} a^2 - k}} = t - t'$$

using an hyperbolic substitution

$$\sqrt{\frac{\alpha l^2}{2}} \operatorname{arsinh} \left( \sqrt{-\frac{2}{\alpha l^2 k}} a \right) = t - t'.$$

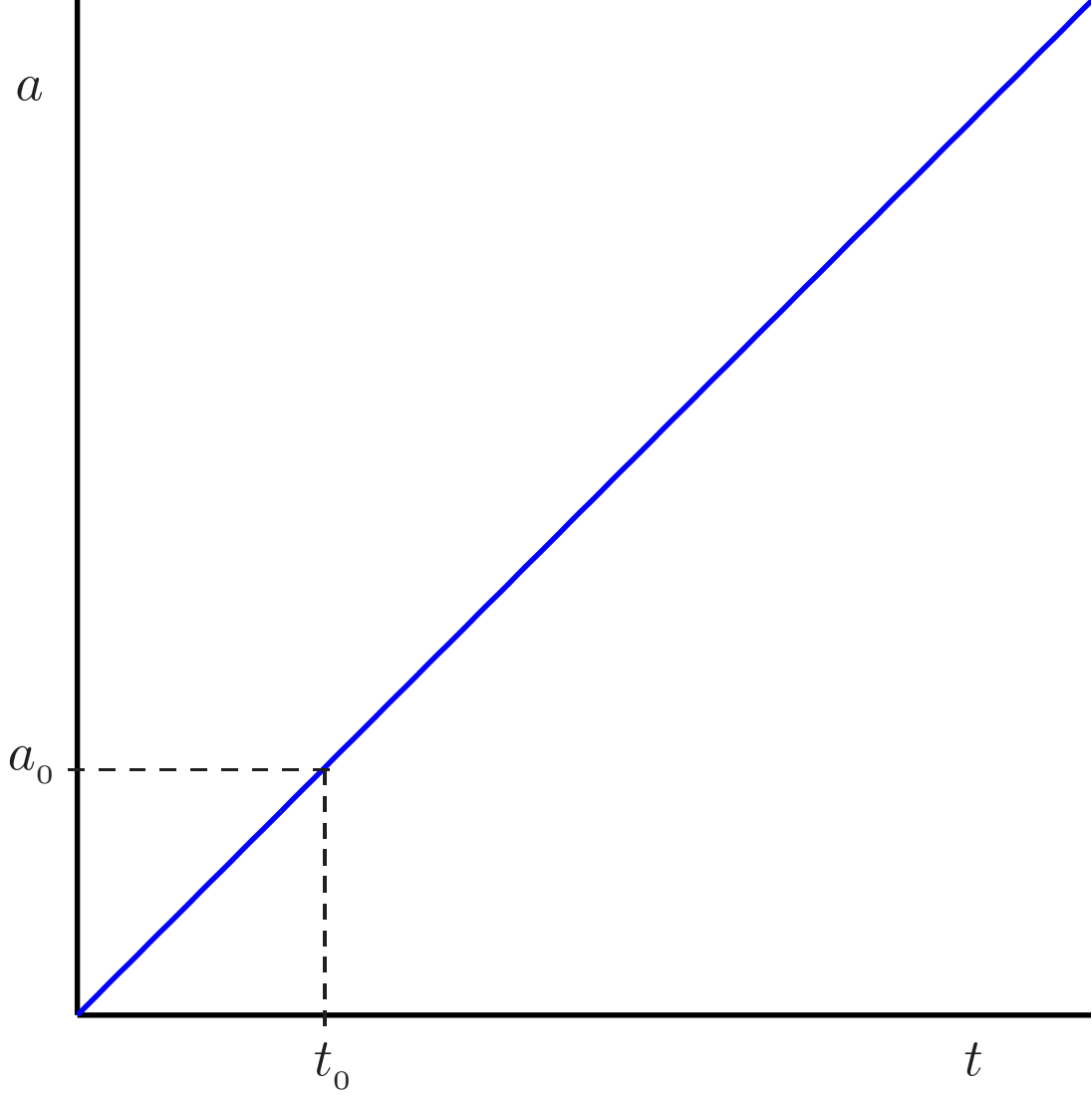


FIG. 1. Graph of  $a(t) = \sqrt{-k}(t - t_0) + a_0$  ( $k = -1$ ).

where  $t'$  is a constant of integration, we obtain

$$A = \sqrt{-\frac{\alpha l^2 k}{2}} \quad (54)$$

and therefore

$$a(t) = \sqrt{-\frac{\alpha l^2 k}{2}} \sinh \left( \sqrt{\frac{2}{\alpha l^2}} (t - t') \right), \quad (55)$$

the initial condition  $a_0 = a(t = t_0)$  leads

$$\begin{aligned} a(t) = & \sqrt{-\frac{\alpha l^2 k}{2}} \\ & \times \sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arsinh} \left( \sqrt{-\frac{2}{\alpha l^2 k}} a_0 \right) \right] \end{aligned} \quad (56)$$

and

$$\begin{aligned} \dot{a}(t) &= \sqrt{-k} \\ &\times \cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arsinh} \left( \sqrt{-\frac{2}{\alpha l^2 k}} a_0 \right) \right]. \end{aligned} \quad (57)$$

This results shows that if  $\alpha > 0$ , then there is an accelerated expansion (see Fig. 2).

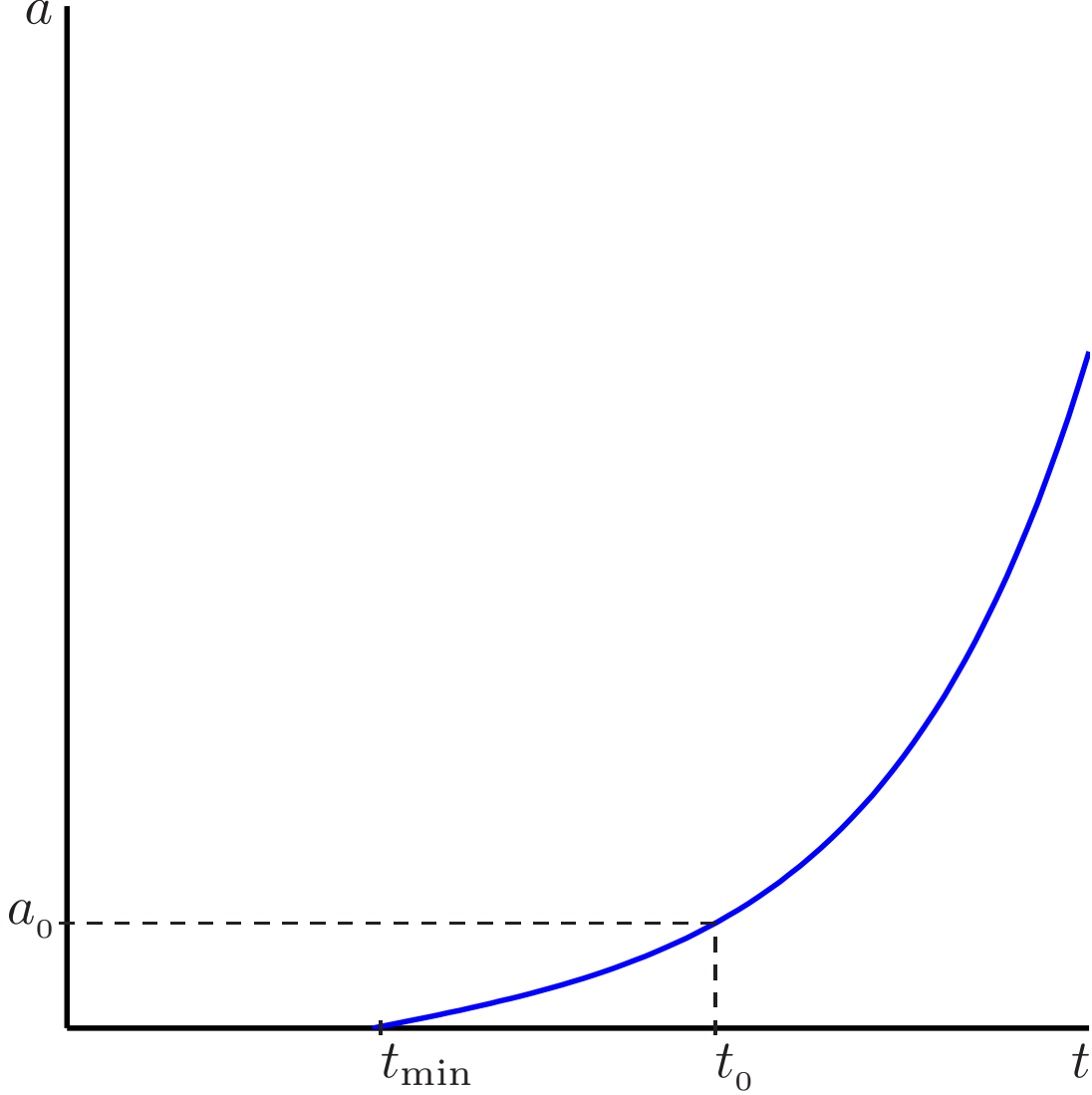


FIG. 2. Graph of  $a(t)$  with  $\alpha > 0$  and  $k = -1$ . See equation (56).

On the another hand, from (56) and (57) we can see that

$$\ddot{a}(t) = \frac{2}{\alpha l^2} a(t), \quad (58)$$

replacing (56), (57) and (58) into (41 - 44) we obtain

$$\rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha l^2}, \quad (59)$$

i.e., we have an accelerated expansion when the energy density is positive and pressure is negative (like a cosmological constant positive).

From equation (45) we find

$$-\frac{\dot{h}}{h - h(0)} = \frac{\dot{a}}{a}. \quad (60)$$

Integrating, we find

$$h(t) = \frac{C}{\sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arsinh} \left( \sqrt{-\frac{2}{\alpha l^2 k}} a_0 \right) \right]} + h(0) \quad (61)$$

where  $C$  is a constant of integration. The initial condition  $h_0 = h(t_0)$  leads

$$h(t) = \frac{(h_0 - h(0)) \sqrt{\frac{2}{\alpha l^2 k}} a_0}{\sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arsinh} \left( \sqrt{-\frac{2}{\alpha l^2 k}} a_0 \right) \right]} + h(0)$$

from where we can see that  $h(t) \rightarrow h(0)$  when  $t \rightarrow \infty$

*b. Case  $\alpha < 0$ :*

Consider now the case when the constant  $\alpha$  is negative. The *ansatz*

$$a(t) = A \sin \left( \sqrt{-\frac{2}{\alpha l^2}} (t - t') \right) \quad (62)$$

with  $t'$  a constant of integration, leads

$$A = \sqrt{\frac{\alpha l^2 k}{2}}, \quad (63)$$

therefore

$$a(t) = \sqrt{\frac{\alpha l^2 k}{2}} \sin \left( \sqrt{-\frac{2}{\alpha l^2}} (t - t') \right). \quad (64)$$

The initial condition  $a_0 = a(t = t_0)$ , leads

$$a(t) = \sqrt{\frac{\alpha l^2 k}{2}} \times \sin \left[ \sqrt{-\frac{2}{\alpha l^2}} (t - t_0) + \arcsin \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right] \quad (65)$$

and

$$\begin{aligned} \dot{a}(t) = & \sqrt{-k} \\ & \times \cos \left[ \sqrt{-\frac{2}{\alpha l^2}}(t - t_0) + \arcsin \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right]. \end{aligned} \quad (66)$$

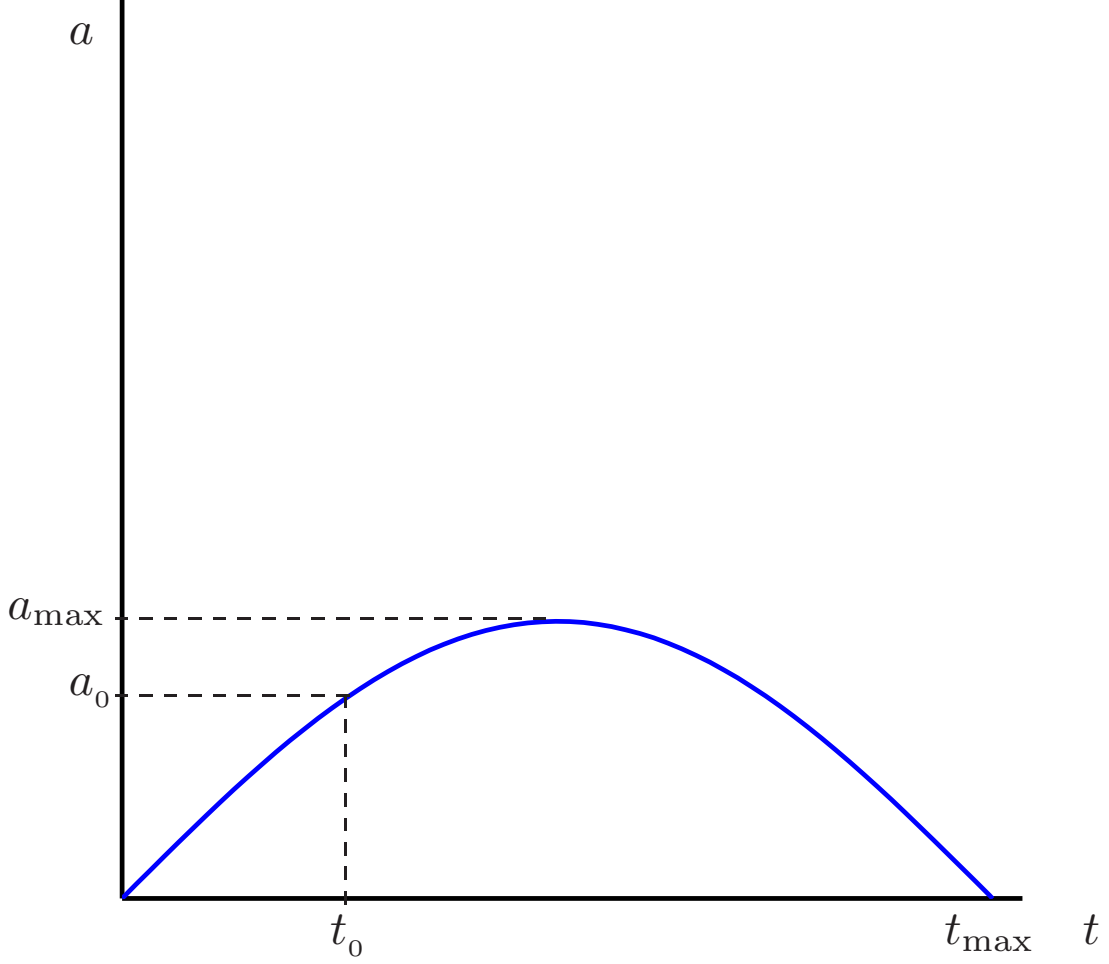


FIG. 3. Graph of  $a(t)$  with  $\alpha < 0$  and  $k = -1$ . See equation (65).

Therefore if  $a(t) > 0$  then  $\ddot{a}(t) < 0$ , which shows that if  $\alpha < 0$ , then there is a decelerated expansion (see Fig. 3).

On the another hand, replacing (65) and (66) into (41 - 44) we obtain

$$\rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha l^2}. \quad (67)$$

Since the energy momentum tensor is given by

$$\tilde{T}_{\mu\nu} = \alpha T_{\mu\nu}^{(h)} = \text{diag}(\alpha \rho^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}, \alpha p^{(h)}) \quad (68)$$



we have that the corresponding energy density and pressure are ( $\alpha < 0$ )

$$\tilde{\rho} = \alpha \rho^{(h)} = \frac{12}{\kappa_5 \alpha l^2} < 0, \quad (69)$$

$$\tilde{p} = \alpha p^{(h)} = -\frac{12}{\kappa_5 \alpha l^2} > 0, \quad (70)$$

i.e., the energy density is negative and the pressure is positive (like a cosmological constant negative).

From equation (45) we find

$$-\frac{\dot{h}}{h - h(0)} = \frac{\dot{a}}{a}. \quad (71)$$

Integrating, we find

$$h(t) = \frac{C}{\sin \left[ \sqrt{-\frac{2}{\alpha l^2}}(t - t_0) + \arcsin \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right]} + h(0) \quad (72)$$

where  $C$  is a constant of integration. The initial condition  $h_0 = h(t_0)$ , leads

$$h(t) = \frac{(h_0 - h(0)) \sqrt{\frac{2}{\alpha l^2 k}} a_0}{\sin \left[ \sqrt{-\frac{2}{\alpha l^2}}(t - t_0) + \arcsin \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right]} + h(0). \quad (73)$$

## B. Case $T_{\mu\nu} = 0$ and $k = 0$

Introducing (43) into (41) and considering  $k = 0$ , we obtain

$$6 \left( \frac{\dot{a}}{a} \right)^2 = 3l^2 \alpha \left( \frac{\dot{a}}{a} \right)^4 \quad (74)$$

which can be rewritten as

$$\left( \frac{\dot{a}}{a} \right)^2 \left( \frac{2}{\alpha l^2} - \frac{\dot{a}^2}{a^2} \right) = 0. \quad (75)$$

### 1. Static solution $\dot{a} = 0$

The solution for an static universe is given by

$$a(t) = a_0 \quad (76)$$

which leads

$$\rho^{(h)} = p^{(h)} = 0 \quad (77)$$

and the equation (45) is satisfied for all  $h(t)$ .

## 2. Non-static solution $\dot{a} \neq 0$

From (75) we obtain

$$\dot{a}^2 - \frac{2}{\alpha l^2} a^2 = 0. \quad (78)$$

This equation have solution, *only if*  $\alpha > 0$ .

### a. Case $\alpha > 0$ :

In this case we have an expanding universe

$$a(t) = A \exp \left( \sqrt{\frac{2}{\alpha l^2}} t \right). \quad (79)$$

The initial condition  $a_0 = a(t_0)$  leads

$$a(t) = a_0 \exp \left( \sqrt{\frac{2}{\alpha l^2}} (t - t_0) \right) \quad (80)$$

and

$$\rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha l^2}.$$

Replacing (80) into equation (45), solving for  $h(t)$  and using the initial condition  $h_0 = h(t_0)$ , we find

$$h(t) = \frac{h_0 - h(0)}{\exp \left( \sqrt{\frac{2}{\alpha l^2}} (t - t_0) \right)} + h(0). \quad (81)$$

### b. Case $\alpha < 0$ :

In this case it is not possible to find a solution.

## C. Case $T_{\mu\nu} = 0$ and $k = 1$

Introducing (43) into (41) we obtain

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = 3l^2 \alpha \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 \quad (82)$$

which can be rewritten as

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left( \frac{2}{\alpha l^2} - \frac{\dot{a}^2 + k}{a^2} \right) = 0. \quad (83)$$

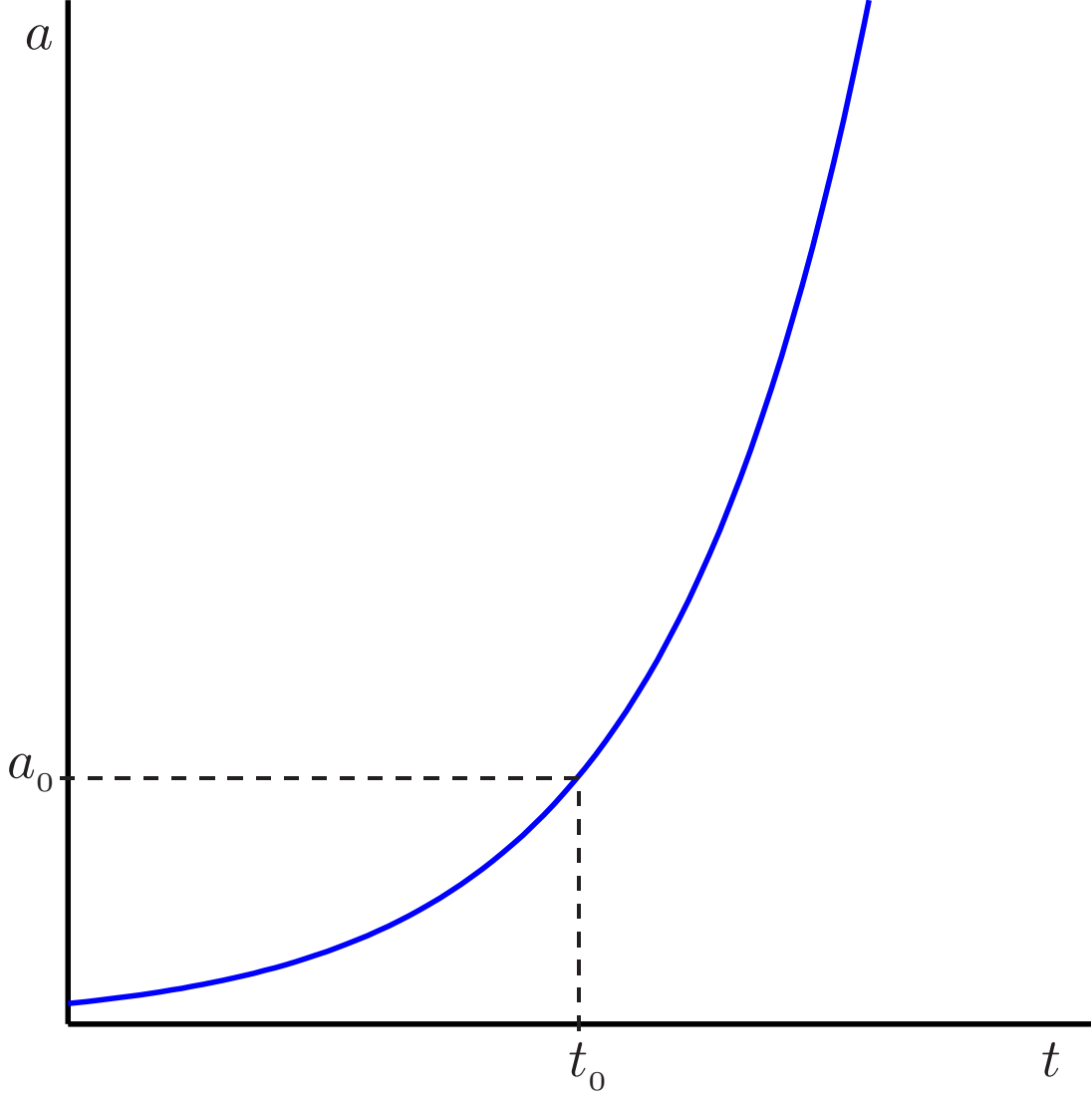


FIG. 4. Graph of  $a(t) = a_0 \exp\left(\sqrt{\frac{2}{\alpha l^2}}(t - t_0)\right)$

1. Case  $\ddot{a} = 0$

In this case it is not possible to find a solution.

2. Case  $\ddot{a} \neq 0$

From equation (83) we obtain

$$\frac{2}{\alpha l^2} a^2 - \dot{a}^2 = k. \quad (84)$$

From (84) we can see two cases:

a. *Case  $\alpha > 0$  :*

If  $\alpha > 0$  we can postulate a solution given by

$$a(t) = A \cosh \left( \sqrt{\frac{2}{\alpha l^2}} (t - t') \right) \quad (85)$$

where  $t'$  is a constant of integration, which leads

$$A = \sqrt{\frac{\alpha l^2 k}{2}}. \quad (86)$$

The initial condition  $a_0 = a(t = t_0)$  leads

$$a(t) = \sqrt{\frac{\alpha l^2 k}{2}} \cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t') + \operatorname{arcosh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right] \quad (87)$$

and

$$\dot{a}(t) = \sqrt{k} \sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arcosh} \left( \sqrt{\frac{2}{\alpha l^2}} a_0 \right) \right] \quad (88)$$

which shows an accelerated expansion (see Fig. 5)

Replacing (87) and (88) into (41 - 44) we obtain

$$\rho^{(h)} = -p^{(h)} = \frac{12}{\kappa_5 \alpha l^2}$$

i.e., we have an accelerated expansion when the energy density is positive and pressure is negative (like a cosmological constant positive)

From equation (45) we find

$$-\frac{\dot{h}}{h - h(0)} = \frac{\dot{a}}{a}, \quad (89)$$

so that

$$h(t) = \frac{C}{\cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arcosh} \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right]} + h(0) \quad (90)$$

where  $C$  is a constant of integration. The initial condition  $h_0 = h(t_0)$  leads

$$h(t) = \frac{(h_0 - h(0)) \sqrt{\frac{2}{\alpha l^2 k}} a_0}{\cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arcosh} \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right]} + h(0)$$

from where we can see that  $h(t) \rightarrow h(0)$  when  $t \rightarrow \infty$ .

b. *Case  $\alpha < 0$  :*

If  $\alpha < 0$  the equation (84) have no solution.

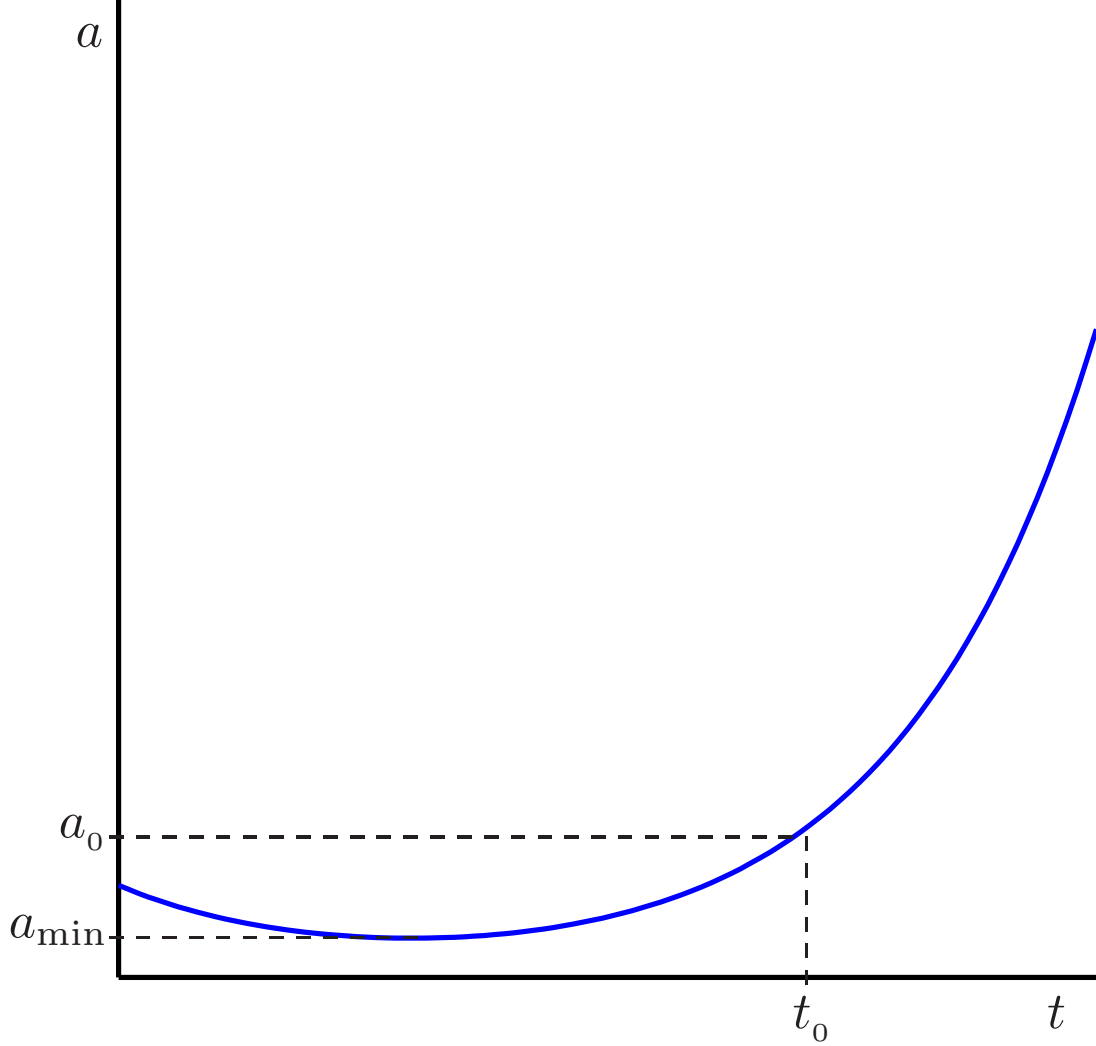


FIG. 5. Graph of  $a(t)$  with  $\alpha > 0$  and  $k = 1$ . See equation (87).

#### D. Era of Dark Energy from Einstein-Chern-Simons gravity

The results in the previous section are summarized in Tables I, II and III.

So that we have found solutions that describe accelerated expansion for the three possible cosmological models of the universe. Namely, spherical expansion ( $k = 1$ ), flat expansion ( $k = 0$ ) and hyperbolic expansion ( $k = -1$ ) when the constant  $\alpha$  is greater than zero. This means that the Einstein-Chern-Simons field equations have as a of their solutions a universe in accelerated expansion. This result allow us to conjeture that this solutions are compat-ible with the era of Dark Energy and that the energy-momentum tensor for the field  $h^a$  corresponds to a form of positive cosmological constant.

From this solutions we can see that as time passes, the  $h(t)$  decreases rapidly to  $h(0)$ , a

TABLE I. Solutions for scale factor of an open space  $k = -1$  (hyperbolic).

Dynamics	$\alpha$	$\rho^{(h)}$	$p^{(h)}$	$\Lambda$
$a(t)$				compatible
Accelerated	$> 0$	$> 0$	$< 0$	$> 0$
Decelerated	$< 0$	$< 0$	$> 0$	$< 0$
No accelerated ( <i>Vacuum</i> )	any	0	0	—

 TABLE II. Solutions for scale factor of a flat space  $k = 0$ .

Dynamics	$\alpha$	$\rho^{(h)}$	$p^{(h)}$	$\Lambda$
$a(t)$				compatible
Accelerated	$> 0$	$> 0$	$< 0$	$> 0$
Stationary ( <i>Vacuum</i> )	any	0	0	—

constant value, keeping constant matter density.

We have also shown that the EChS field equations have solutions that allows us to identify the energy-momentum tensor for the field  $h^a$  with a negative cosmological constant.

## VI. CONSISTENCY OF THE SOLUTIONS WITH THE "ERA OF MATTER"

In the previous section, we find that the solutions of EChS field equations, with  $T_{\mu\nu} = 0$ , can be useful as models of the era of Dark Energy. In this section we review the consistency of this equations with the era of Matter.

We will consider the ordinary matter as dust ( $\rho \neq 0$ ,  $p = 0$ ), such as occurs in standard cosmology. The non-ordinary matter will be modeled as a perfect fluid ( $\rho^{(h)} \neq 0$  y  $p^{(h)} \neq 0$ ). In this case the field equations (35 - 39) takes the form

TABLE III. Solutions for scale factor of a closed space  $k = 1$ .

Dynamics	$\alpha$	$\rho^{(h)}$	$p^{(h)}$	$\Lambda$
$a(t)$				compatible
Accelerated	$> 0$	$> 0$	$< 0$	$> 0$

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 (\rho + \alpha \rho^{(h)}) \quad (91)$$

$$3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \alpha p^{(h)} \quad (92)$$

$$\frac{3l^2}{\kappa_5} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \rho^{(h)} \quad (93)$$

$$\frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)} \quad (94)$$

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (h - h(0)) \frac{\dot{a}}{a} + \dot{h} \right] = 0 \quad (95)$$

and the conservation equations (34) (*divergence-free energy-momentum tensor*) for each fluids are given by

$$\dot{\rho} + 4 \frac{\dot{a}}{a} \rho = 0 \quad (96)$$

and

$$\dot{\rho}^{(h)} + 4 \frac{\dot{a}}{a} (\rho^{(h)} + p^{(h)}) = 0. \quad (97)$$

The equation (96) have as solution

$$\rho(t) = \left( \frac{a_0}{a(t)} \right)^4 \rho_0 \quad (98)$$

where the initial conditions  $a_0 = a(t_0)$  and  $\rho_0 = \rho(t_0)$  has been set.

Replacing (98) and (93) into equation (91) we have

$$\left( \frac{\dot{a}^2 + k}{a^2} \right)^2 - 2A \left( \frac{\dot{a}^2 + k}{a^2} \right) + AB \frac{a_0^4}{a^4} = 0 \quad (99)$$

where we defined

$$A := \frac{1}{\alpha l^2} \quad , \quad B := \frac{\kappa_5 \rho_0}{3}. \quad (100)$$

**A. Case  $k = -1$**

In this case, the equation (99) can be rewritten

$$\left(\frac{\dot{a}^2 - 1}{a^2}\right)^2 - 2A\left(\frac{\dot{a}^2 - 1}{a^2}\right) + AB\frac{a_0^4}{a^4} = 0 \quad (101)$$

where we find

$$\dot{a} = \pm \sqrt{Aa^2 \left(1 \pm \text{sgn}(A) \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}}\right) + 1}. \quad (102)$$

*1. Case  $\alpha > 0$*

In this case

$$A = \frac{1}{\alpha l^2} > 0. \quad (103)$$

From (102) we can see that  $\dot{a}$  is well defined if

$$a \geq a_{\min} = \sqrt[4]{\frac{B}{A}} a_0 \quad (104)$$

where

$$a_{\min} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0.$$

On the other hand  $a_0$  must satisfy

$$a_0 \geq a_{\min} \quad (105)$$

so that

$$\frac{B}{A} \leq 1 \quad \text{i.e.,} \quad B \leq A \quad (106)$$

and therefore

$$\rho_0 \leq \rho_{\max} = \frac{3}{\kappa_5 \alpha l^2} \quad (107)$$

These results allow us to analyze the radicand in (102)

$$Aa^2 \left(1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}}\right) + 1 \geq 0, \quad (108)$$

i.e.,

$$-A^2 a_{\min}^4 \leq 1 + 2Aa^2 \quad (109)$$

which is satisfied for all  $a$ .



*a. Plus or minus sign?*

The choice of the sign into the radicand has information about the allowed values of  $\dot{a}$ . Let us consider  $\dot{a} > 0$  (the analysis of the case  $\dot{a} < 0$  is very similar)

$$\dot{a} = \sqrt{Aa^2 \left( 1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}} \right) - k}. \quad (110)$$

The function  $\dot{a}(a)$  is monotonically increasing (decreasing) if we consider the plus (minus) sign in front of the square root.

From (110) we can see that there exist  $\dot{a}_{\text{cri}}$

$$\dot{a}_{\text{cri}} := \dot{a}(a_{\min}) = \sqrt{\sqrt{\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0^2 - k}. \quad (111)$$

If we consider the plus (minus) sign in front of the square root,  $\dot{a}_{\text{cri}}$  is the minimum (maximum) value of  $\dot{a}$ .

If there is a limit to  $a \gg a_{\min}$ , then

$$\begin{aligned} \dot{a} &= \pm \sqrt{Aa^2 \left( 1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}} \right) - k} \\ &\approx \pm \sqrt{Aa^2 \left\{ 1 \pm \left( 1 - \frac{a_{\min}^4}{2a^4} \right) \right\} - k} \end{aligned} \quad (112)$$

where  $k = -1$ .

*b. Case where the sign is “+”*

In this case

$$\dot{a} = \pm \sqrt{Aa^2 \left( 2 - \frac{a_{\min}^4}{2a^4} \right) - k} \approx \pm \sqrt{2Aa^2 - k}, \quad (113)$$

whose approximate solution is

$$\begin{aligned} a(t) &= \pm \sqrt{-\frac{\alpha l^2 k}{2}} \\ &\times \sinh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \text{arsinh} \left( \sqrt{-\frac{2}{\alpha l^2 k}} a_0 \right) \right] \end{aligned} \quad (114)$$

where we use  $A = \frac{1}{\alpha l^2}$  and  $k = -1$ .

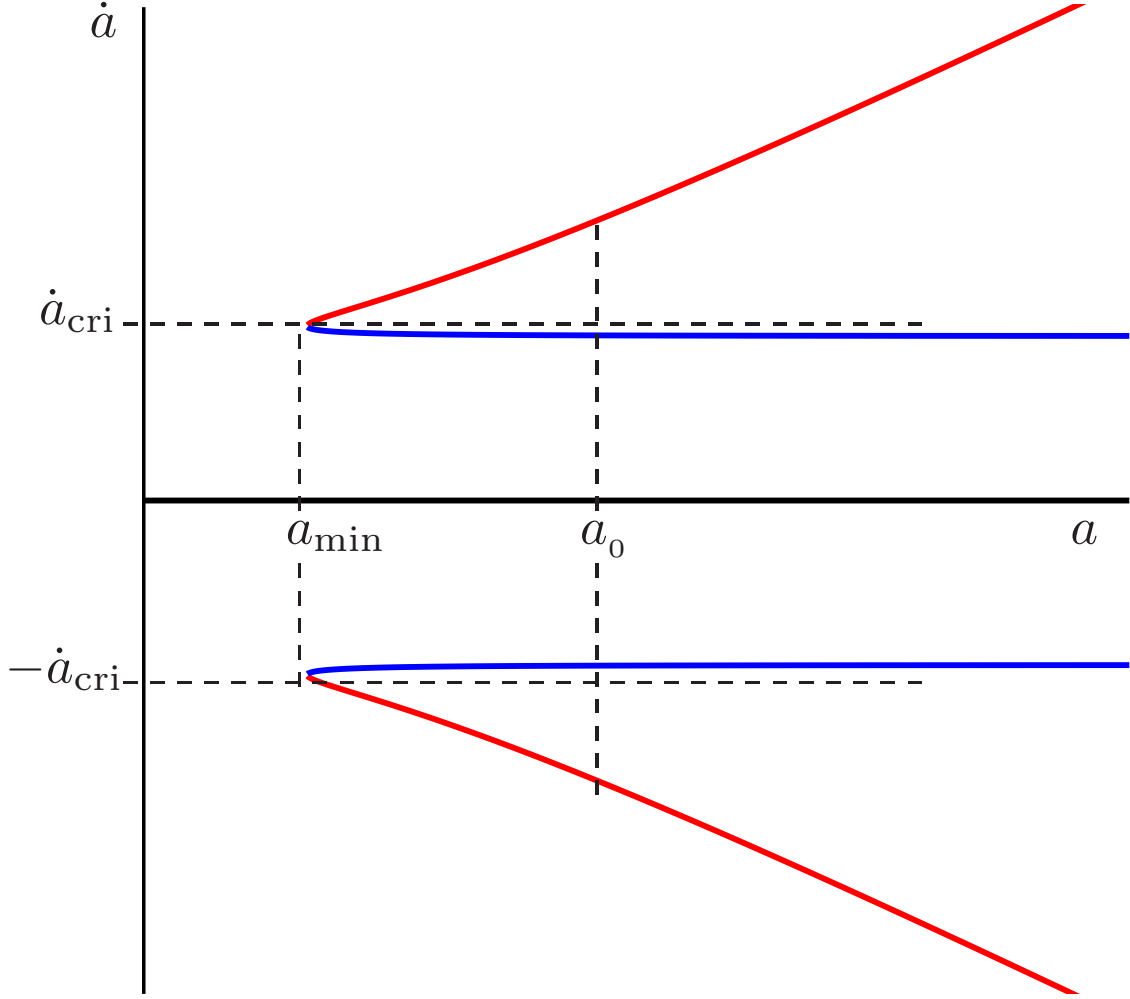


FIG. 6. For every  $a_0$  allowed there are two different values for  $\dot{a} > 0$ : evolution with  $\dot{a}$  approximate constant and evolution accelerated(decelerated).

*c. Case where the sign is “-”*

In this case

$$\dot{a} = \pm \sqrt{A \frac{a_{\min}^4}{2a^2} - k} \approx \pm \sqrt{-k} \quad (115)$$

whose approximate solution is

$$a(t) = \pm \sqrt{-k}(t - t_0) + a_0$$

where we use  $k = -1$ .

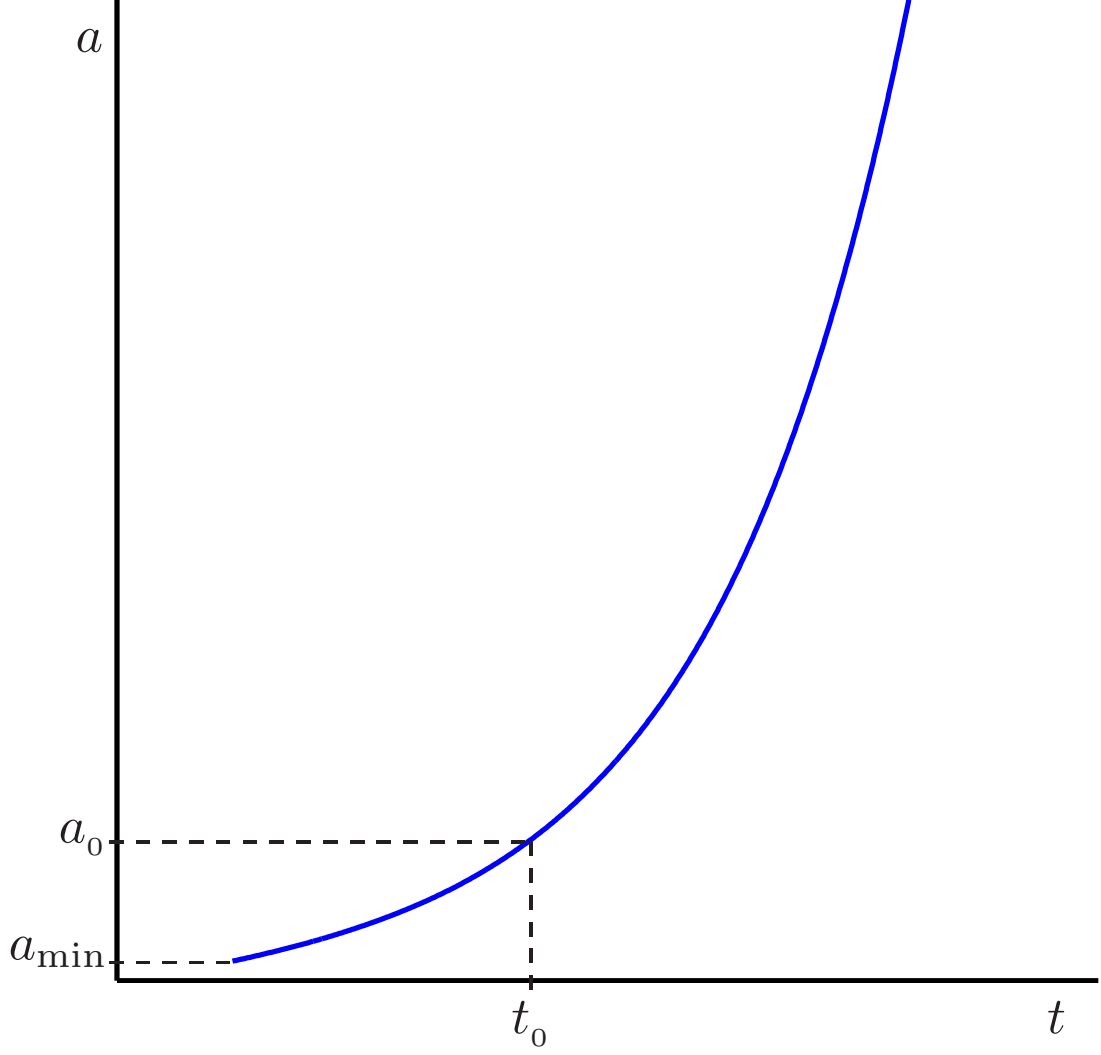


FIG. 7. Numerical solution with  $A > 0$ ,  $k = -1$  and  $\dot{a}_0 > \dot{a}_{\text{cri}}$  of  $\dot{a} = \sqrt{Aa^2 \left( 1 + \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right)} - k$ .

## 2. Case $\alpha < 0$

In this case

$$A = \frac{1}{\alpha l^2} < 0. \quad (116)$$

From (102) we can see that  $\dot{a}$  is well defined if

$$1 - \frac{B}{A} \frac{a_0^4}{a^4} \geq 0, \quad (117)$$

but this condition is satisfied for all  $a$ .

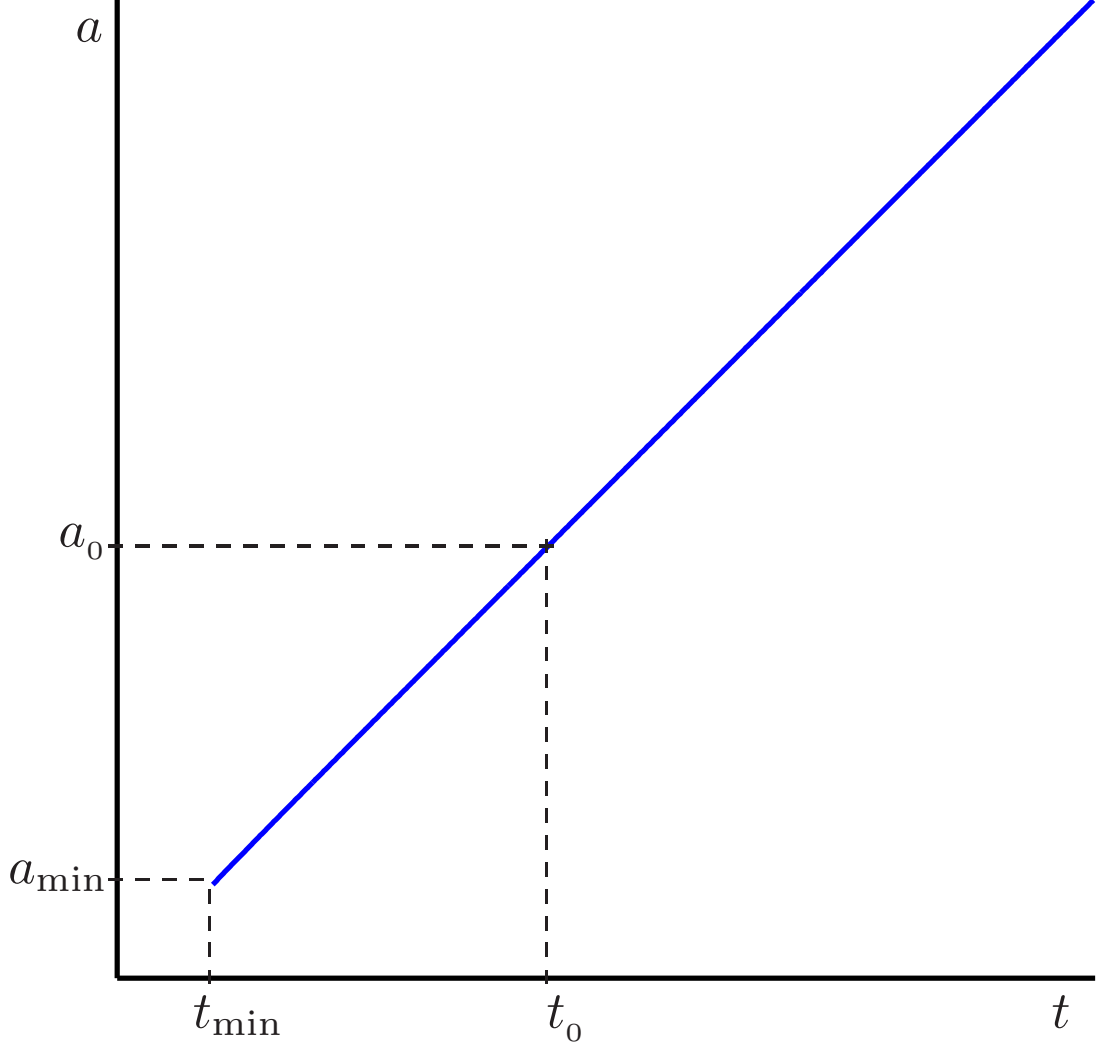


FIG. 8. Numerical solution with  $A > 0$ ,  $k = -1$  and  $\dot{a}_0 < \dot{a}_{\text{cri}}$  of  $\dot{a} = \sqrt{Aa^2 \left( 1 + \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right)} - k$ .

*a. Case where the sign is ‘+’*

In this case

$$Aa^2 \left( 1 + \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right) - k \geq 0, \quad (118)$$

so that

$$\frac{k - Aa^2}{Aa^2} \geq \sqrt{1 - \frac{Ba_0^4}{Aa^4}}. \quad (119)$$

The left side of the last equation must be positive, i.e.,

$$k - Aa^2 \leq 0 \quad \text{or} \quad a \leq \sqrt{\frac{k}{A}}. \quad (120)$$

From (119) we obtain

$$k^2 - 2Aka^2 \geq -ABa_0^4 \quad (121)$$

and again, the left side of the last equation must be positive, i.e.,

$$k^2 - 2Aka^2 \geq 0 \quad \Longleftrightarrow \quad a \leq \sqrt{\frac{k}{2A}} \quad (122)$$

and from (121) we find

$$a \leq \sqrt{\frac{k^2 + ABa_0^4}{2Ak}}. \quad (123)$$

Since

$$\sqrt{\frac{k}{A}} > \sqrt{\frac{k}{2A}} > \sqrt{\frac{k^2 + ABa_0^4}{2Ak}} = a_{\max} \geq a, \quad (124)$$

we have found a maximum value for  $a$

$$a_{\max} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}} \quad (125)$$

and therefore

$$\dot{a}(a = a_{\max}) = 0 \quad (126)$$

i. e.,  $a_{\max}$  is a local maximum. It is direct to prove that  $\dot{a} \neq 0$  for  $a \neq a_{\max}$ . If  $a$  has a maximum value  $a_{\max}$  then (see (98))

$$\rho(t) = \left(\frac{a_0}{a(t)}\right)^4 \rho_0 \geq \left(\frac{a_0}{a_{\max}}\right)^4 \rho_0 = \rho_{\min}. \quad (127)$$

This means that  $\rho$  has a minimum value  $\rho_{\min}$  given by

$$\rho_{\min} = \left(\frac{6ka_0^2}{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}\right)^2 \rho_0 \quad (128)$$

where  $k = -1$ .

Consider the case where  $\dot{a} > 0$ . We just consider  $\dot{a} > 0$  because the analysis of the case  $\dot{a} < 0$  looks very similar. In this case

$$\dot{a} = \sqrt{Aa^2 \left(1 + \sqrt{1 - \frac{Ba_0^4}{Aa^4}}\right) - k} \quad (129)$$

is a decreasing function. We can see that the minimum value of  $\dot{a}$  is given by

$$\dot{a}_{\min} = \dot{a}(a_{\max}) = 0 \quad (130)$$

and the maximum value of  $\dot{a}$  is given by

$$\dot{a}_{\max} = \dot{a}(a = 0) = \sqrt{-\sqrt{-\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0^2 - k}.$$

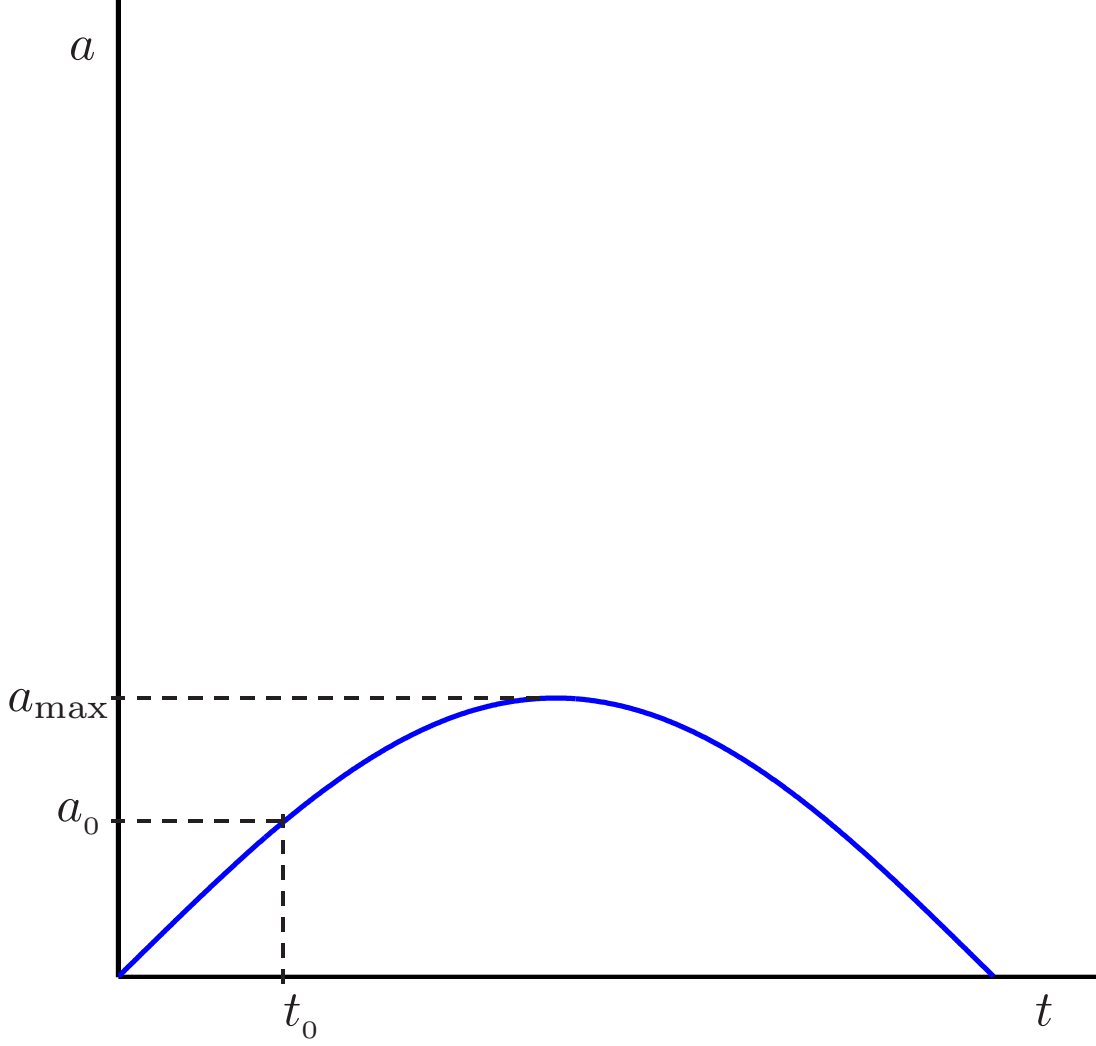


FIG. 9. Solution of  $\dot{a} = \sqrt{Aa^2 \left(1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}}\right)} - k$  with  $A < 0$ ,  $k = -1$  and  $|\dot{a}_0| < \dot{a}_{\max}$ .

*b. Case where the sign is “−”*

In this case we obtain the following condition

$$Aa^2 \left(1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}}\right) - k \geq 0 \quad (131)$$

where  $A = -\frac{1}{\alpha l^2}$  and  $k = -1$ . This condition is trivially satisfied for all  $a$ .

This result implies that  $\dot{a} \neq 0$ . This means that  $a$  has no local maximums/minimums, so  $a$  is monotonically increasing or monotonically decreasing.

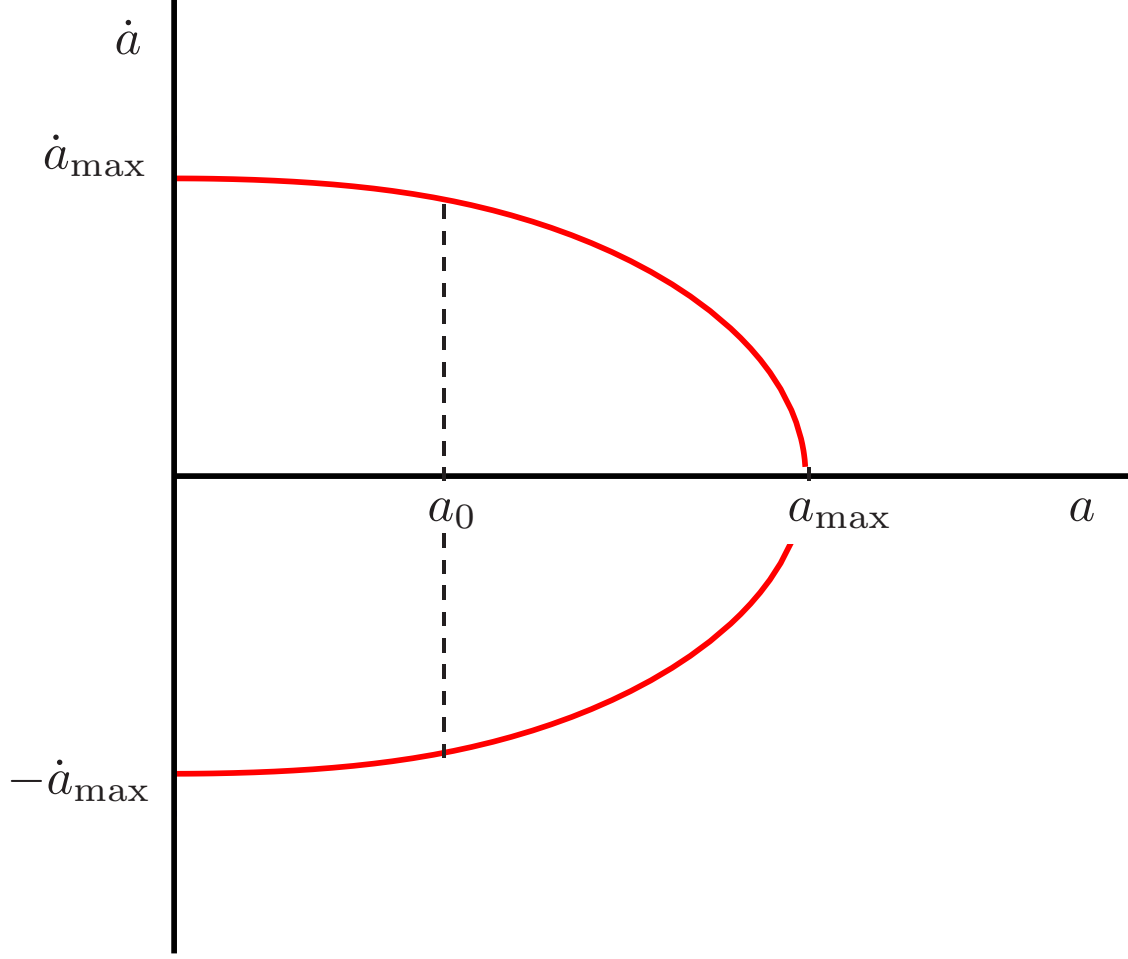


FIG. 10. Phase space for  $A < 0$  and  $k = -1$  with “+” sign.

If there is a limit to  $a \gg \sqrt[4]{-\frac{B}{A}} a_0$ , then

$$\begin{aligned} \dot{a} &= \pm \sqrt{Aa^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right) - k} \\ &\approx \pm \sqrt{Aa^2 \left( 1 - \left( 1 - \frac{Ba_0^4}{2Aa^4} \right) \right) - k} \end{aligned} \quad (132)$$

and

$$\dot{a} = \pm \sqrt{\frac{Ba_0^4}{2a^2} - k} \approx \pm \sqrt{-k}, \quad (133)$$

whose approximate solution is

$$a(t) = \pm \sqrt{-k}(t - t_0) + a_0 \quad (134)$$

where we use  $k = -1$ .

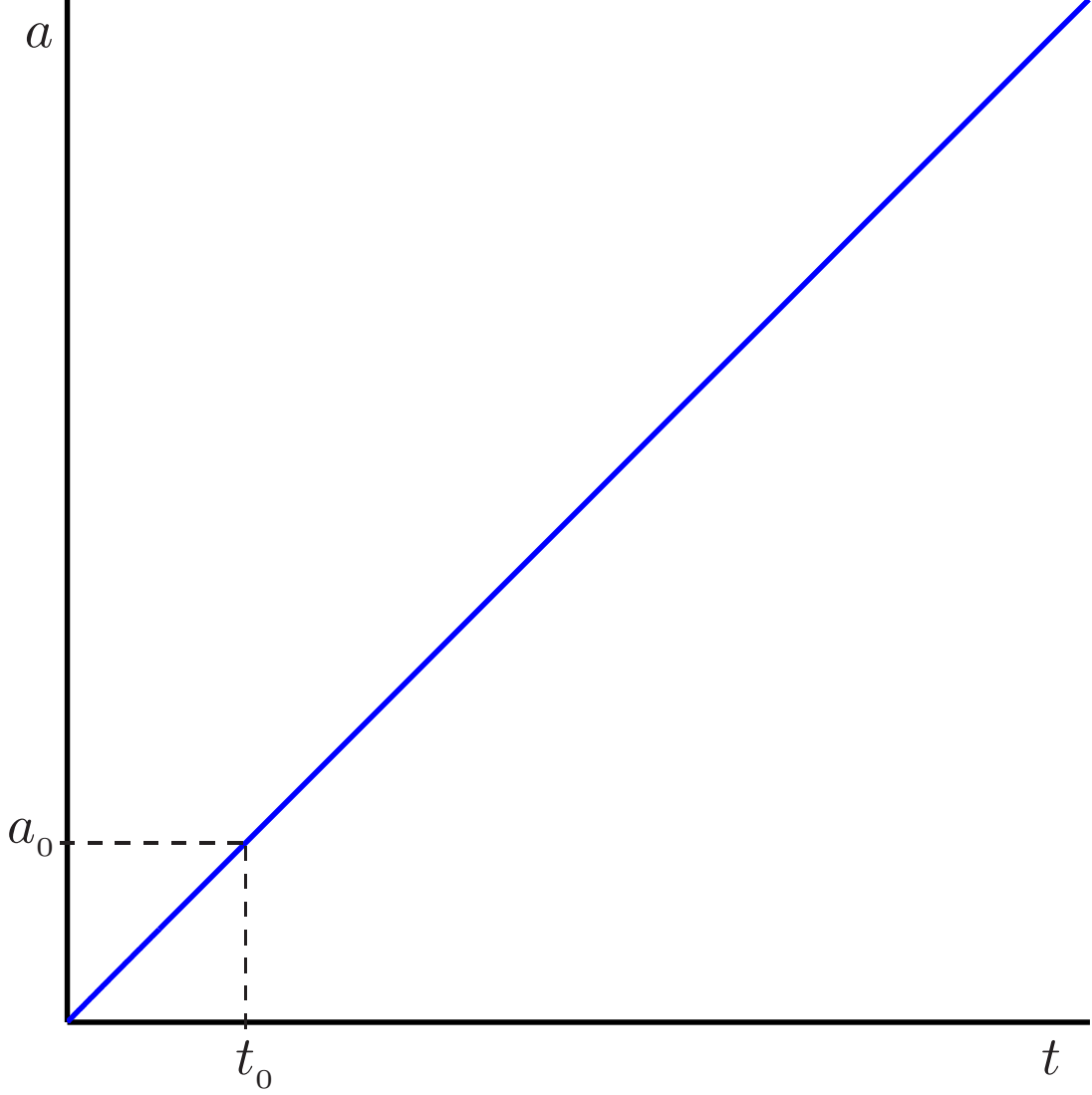


FIG. 11. Solution of  $\dot{a} = \sqrt{Aa^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right)} - k$  with  $A < 0$ ,  $k = -1$  and  $1 < \dot{a}_0 < \dot{a}_{\max}$ .

In this case

$$\dot{a} = \sqrt{Aa^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right)} - k \quad (135)$$

is a decreasing function. The maximum value of  $\dot{a}$  is given by

$$\dot{a}_{\max} = \dot{a}(a = 0) = \sqrt{\sqrt{-\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0^2 - k} \quad (136)$$

and we can see that  $\dot{a}$  tends to a minimum value given by

$$\dot{a}_{\min} = \dot{a}(a \rightarrow \infty) = \sqrt{-k} = 1 \quad (137)$$



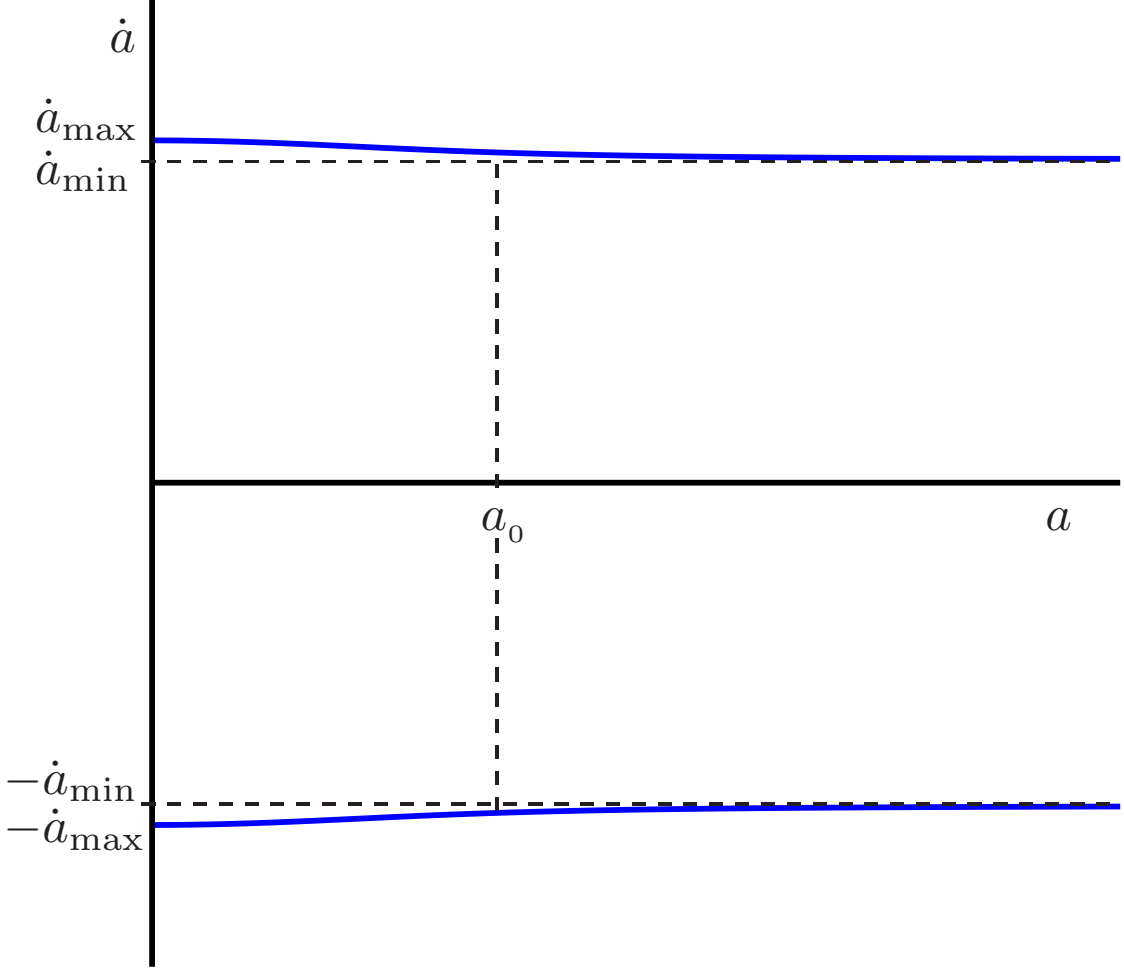


FIG. 12. Phase space for  $A < 0$  and  $k = -1$  with “-” sign.

### B. Case $k = 0$

In this case, the equation (99) takes the form

$$\left(\frac{\dot{a}}{a}\right)^4 - 2A \left(\frac{\dot{a}}{a}\right)^2 + AB \frac{a_0^4}{a^4} = 0 \quad (138)$$

from where

$$\dot{a} = \pm \sqrt{Aa^2 \left( 1 \pm \text{sgn}(A) \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} \right)}. \quad (139)$$

#### 1. Case $\alpha > 0$

In this case

$$A = \frac{1}{\alpha l^2} > 0. \quad (140)$$

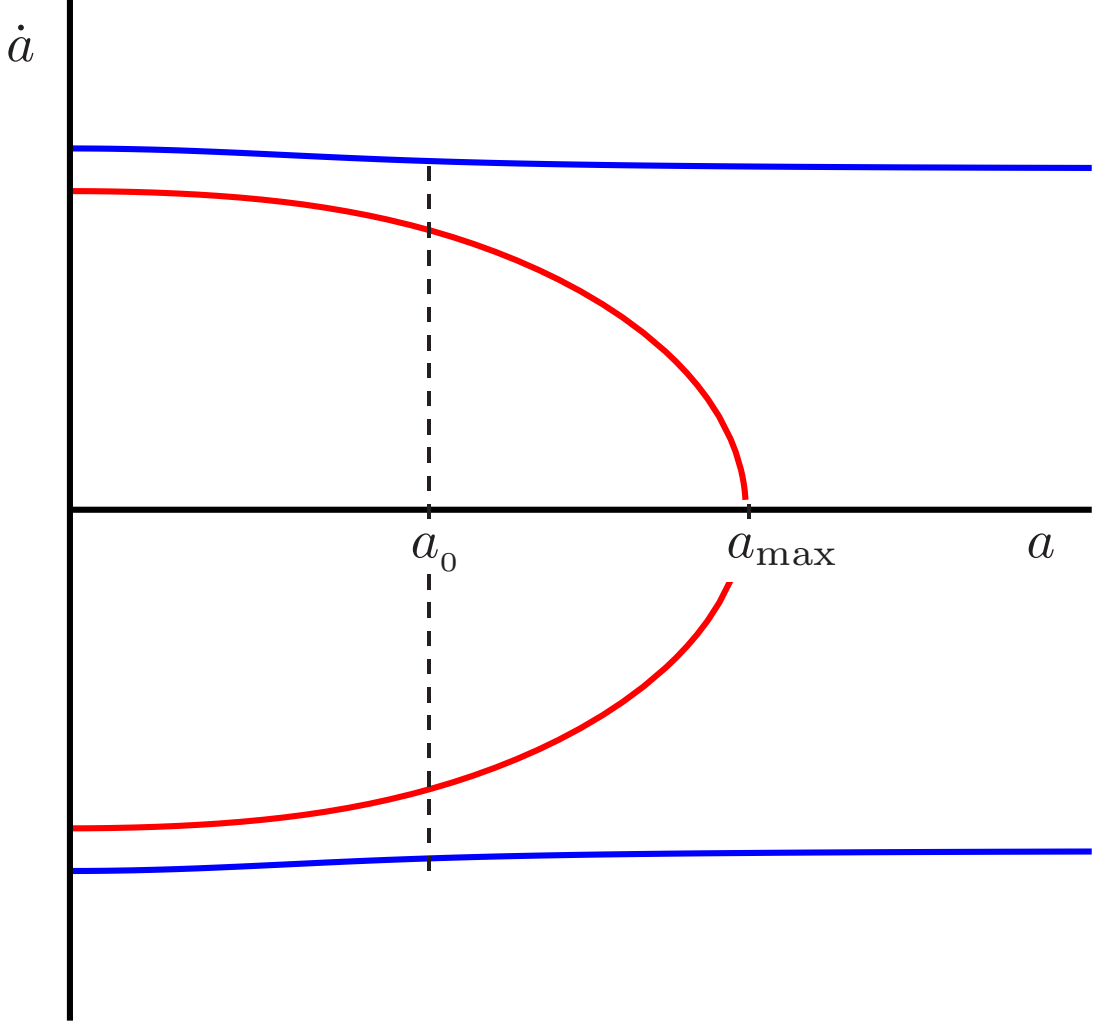


FIG. 13. Phase space for  $A < 0$  and  $k = -1$ . Comparison between phase space with “+” sign (Fig. 10) and “−” sign (Fig. 12).

From (139) we can see that  $\dot{a}$  is well defined if

$$a \geq \sqrt[4]{\frac{B}{A}} a_0 \quad (141)$$

and therefore a minimum value for  $a$  is given by

$$a_{\min} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0. \quad (142)$$

On the other hand  $a_0 \geq a_{\min}$ , so that

$$B \leq A \quad \text{i.e.,} \quad \rho_0 \leq \rho_{\max} = \frac{3}{\kappa_5 \alpha l^2}. \quad (143)$$

These results leads

$$Aa^2 \left( 1 \pm \text{sgn}(A) \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} \right) \geq 0, \quad (144)$$

i.e.,  $a$  has no local maximums/minimums<sup>2</sup>, so that  $a$  is monotonically increasing or monotonically decreasing.

*Plus or minus sign?*

The choice of the sign into the radicand has information about the allowed values of  $\dot{a}$ . Let us consider  $\dot{a} > 0$ , the analysis of the case  $\dot{a} < 0$  is very similar

$$\dot{a} = \sqrt{Aa^2 \left( 1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}} \right)}. \quad (145)$$

The function  $\dot{a}(a)$  is monotonically increasing(decreasing) if we consider the plus(minus) sign in front of the square root.

From (145) we can see that exist  $\dot{a}_{\text{cri}}$

$$\dot{a}_{\text{cri}} := \dot{a}_{\min} = \sqrt{A} a_{\min} = \sqrt[4]{\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0.$$

If we consider the plus (minus) sign in front of the square root,  $\dot{a}_{\text{cri}}$  is the minimum(maximum) value of  $\dot{a}$ .

If there is a limit to  $a \gg a_{\min}$  then

$$\begin{aligned} \dot{a} &= \pm \sqrt{Aa^2 \left( 1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}} \right)} \\ &\approx \pm a \sqrt{A \left( 1 \pm \left( 1 - \frac{a_{\min}^4}{2a^4} \right) \right)} \end{aligned} \quad (146)$$

*a. Case where the sign is “+”*

In this case

$$\dot{a} = \pm a \sqrt{A \left( 2 - \frac{a_{\min}^4}{2a^4} \right)} \approx \pm a \sqrt{2A} \quad (147)$$

---

<sup>2</sup> Only it has a local maximum/minimum if we consider the minus sign into the radicand. In that case the local minimum is  $a_{\min}$ . We can prove that there is no local maximum.

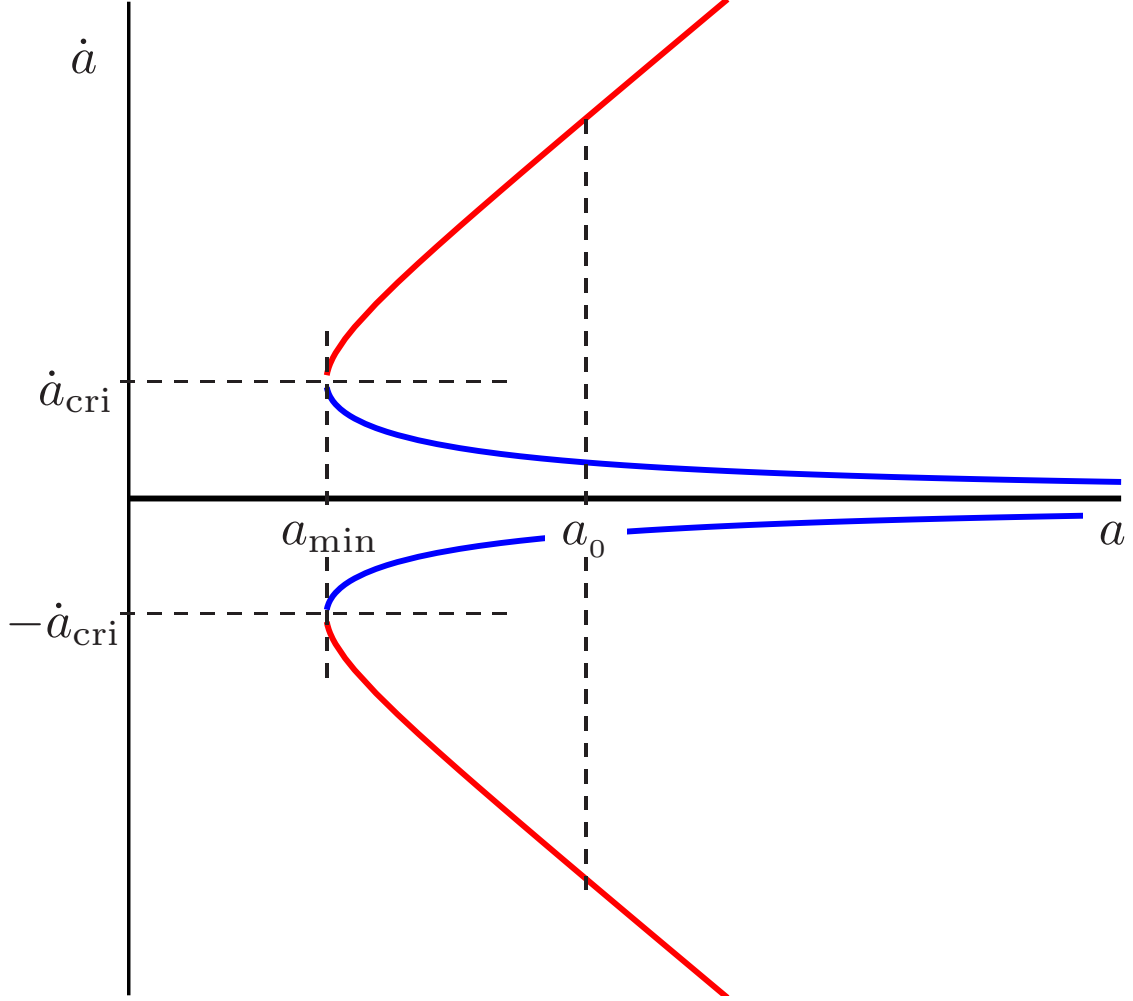


FIG. 14. For every  $a_0$  there are two different values for  $\dot{a}$ : evolution with  $\dot{a} < \dot{a}_{\text{cri}}$  and expansion accelerated(decelerated) with  $|\dot{a}| > |\dot{a}_{\text{cri}}|$ .

whose approximate solution is

$$a(t) = a_0 \exp \left( \pm \sqrt{\frac{2}{\alpha l^2}} (t - t_0) \right) \quad (148)$$

where  $A = \frac{1}{\alpha l^2} > 0$ .

*b. Case where the sign is “-”*

In this case

$$\dot{a} \approx \pm \sqrt{\frac{A}{2}} \frac{a_{\text{min}}^2}{a} \quad (149)$$

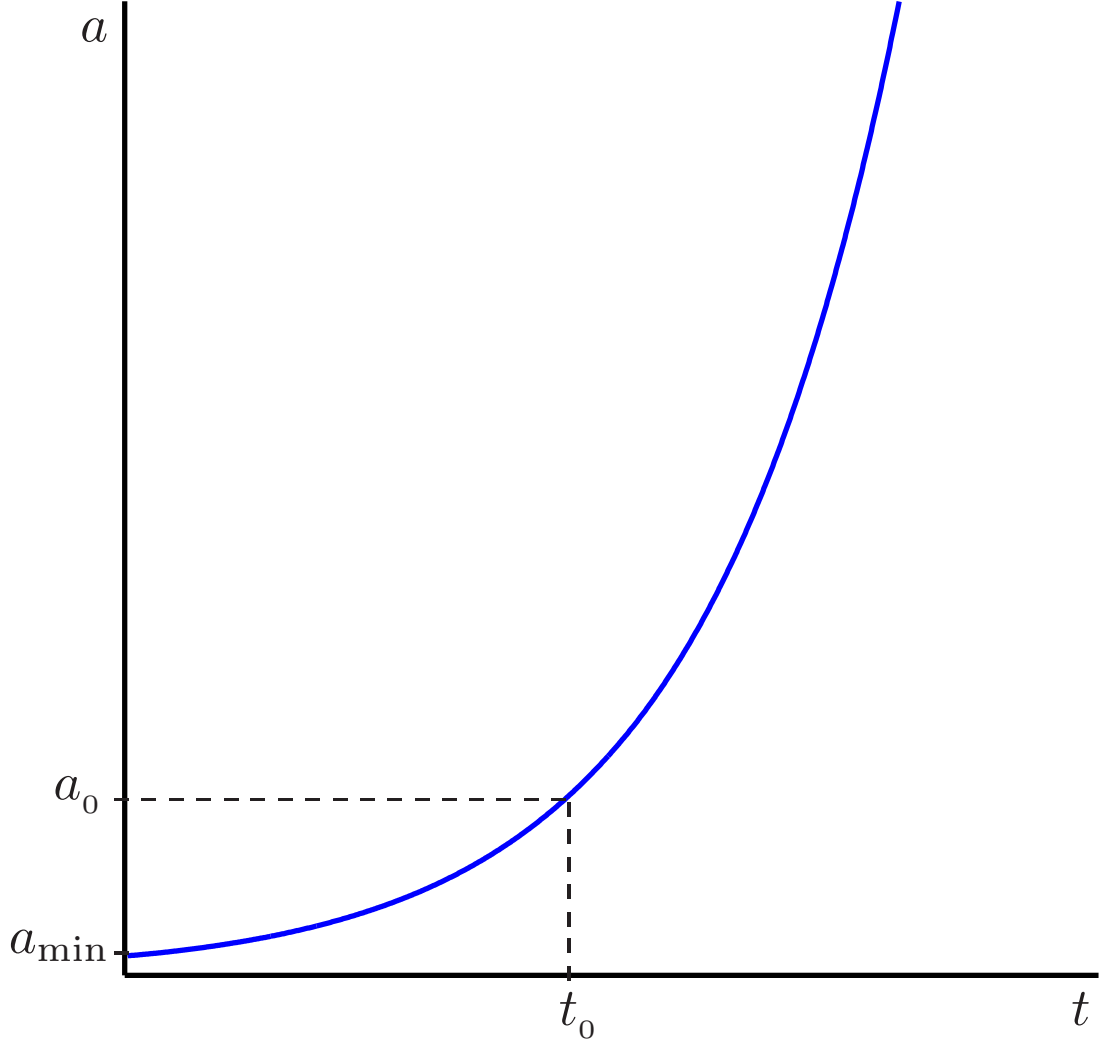


FIG. 15. Solution of  $\dot{a} = \sqrt{Aa^2 \left(1 + \sqrt{1 - \frac{a_{\min}^4}{a^4}}\right)}$  with  $A > 0$  and  $\dot{a}_0 > \dot{a}_{\text{cri}}$ .

whose approximate solution is

$$\begin{aligned}
 a(t) &= \pm \sqrt{a_0^2 \pm \sqrt{\frac{2}{\alpha l^2}} a_{\min}^2 (t - t_0)} \\
 &= \pm a_0 \sqrt{1 \pm \sqrt{\frac{2\kappa_5 \rho_0}{3a_0^4}} (t - t_0)}
 \end{aligned} \tag{150}$$

where we use  $A = \frac{1}{\alpha l^2} > 0$  and  $a_{\min} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0$ .

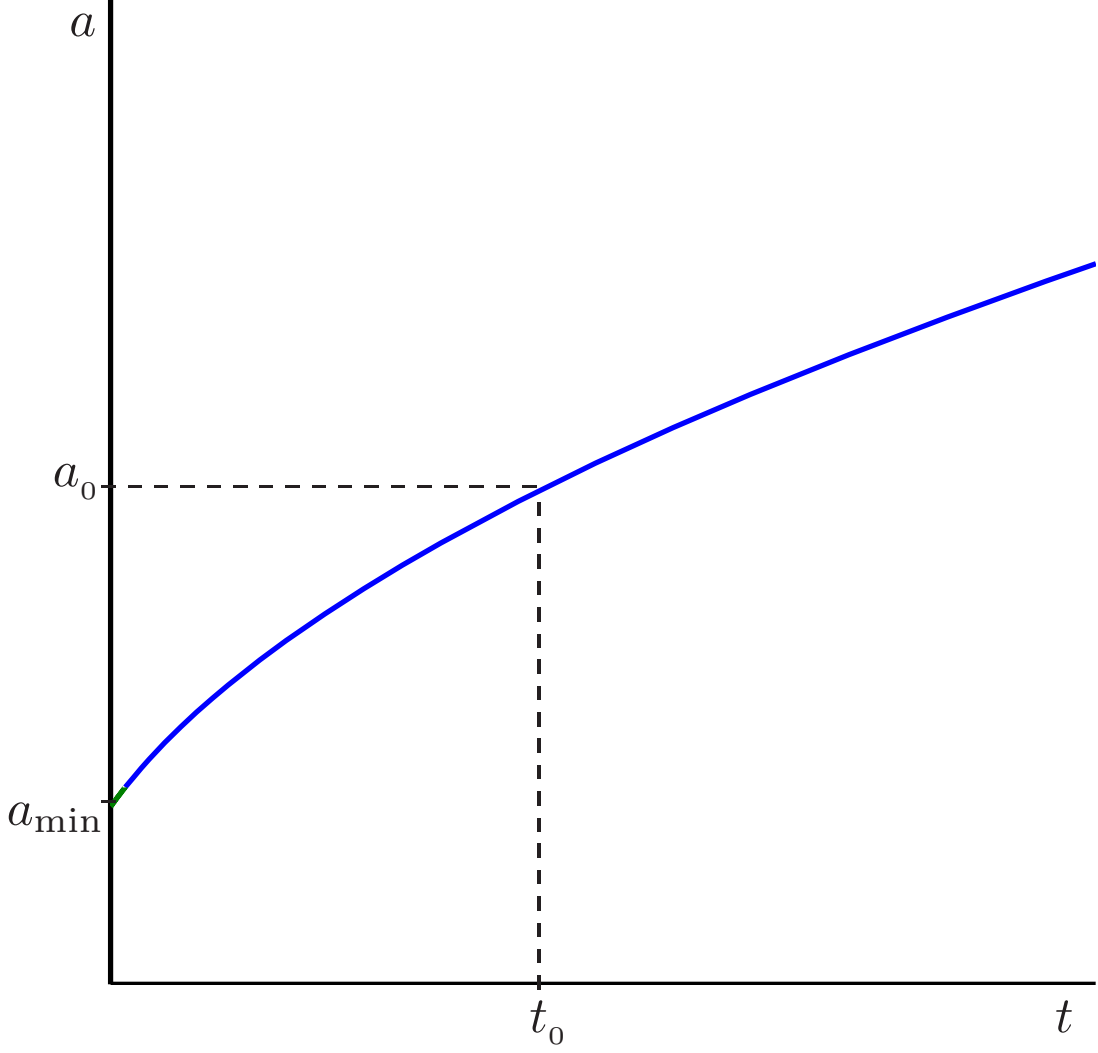


FIG. 16. Solution of  $\dot{a} = \sqrt{Aa^2 \left(1 - \sqrt{1 - \frac{a_{\min}^4}{a^4}}\right)}$  with  $A > 0$  and  $\dot{a}_0 < \dot{a}_{\text{cri}}$ .

## 2. Case $\alpha < 0$

In this case

$$A = \frac{1}{\alpha l^2} < 0 \quad (151)$$

From (139) we can see that  $\dot{a}$  is well defined if

$$1 \mp \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} \leq 0. \quad (152)$$

This condition is only satisfied if we use the minus sign “−” for all  $a$ , i.e.,

$$1 - \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} < 0 \quad (153)$$

and therefore  $a$  has no local maximums/minimums, so  $a$  is monotonically increasing or monotonically decreasing. So that  $\dot{a}$  has a maximum value in  $a = 0$ , i.e.,

$$\dot{a}_{\max} = \dot{a}(a = 0) = \sqrt[4]{-\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0 \quad (154)$$

and  $\dot{a}$  tends to a minimum value given by

$$\dot{a}_{\min} = \dot{a}(a \rightarrow \infty) = 0. \quad (155)$$

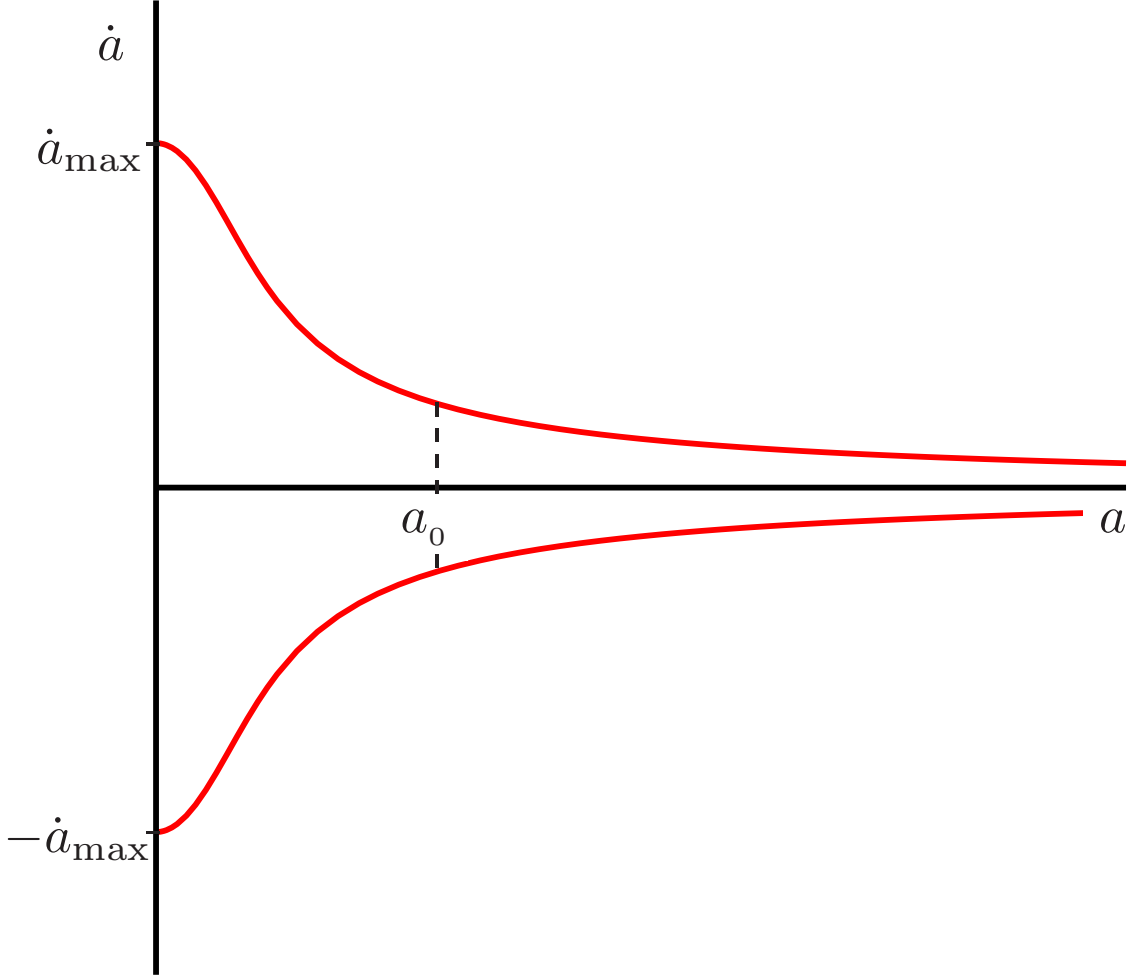


FIG. 17. Phase space for  $A < 0$  and  $k = 0$  with “-” sign.

If exist a limit for  $a \gg \sqrt[4]{-\frac{B}{A}} a_0$  then

$$\dot{a} = \pm \sqrt{A a^2 \left( 1 - \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} \right)} \approx \pm \sqrt{\frac{B}{2}} \frac{a_0^2}{a}, \quad (156)$$

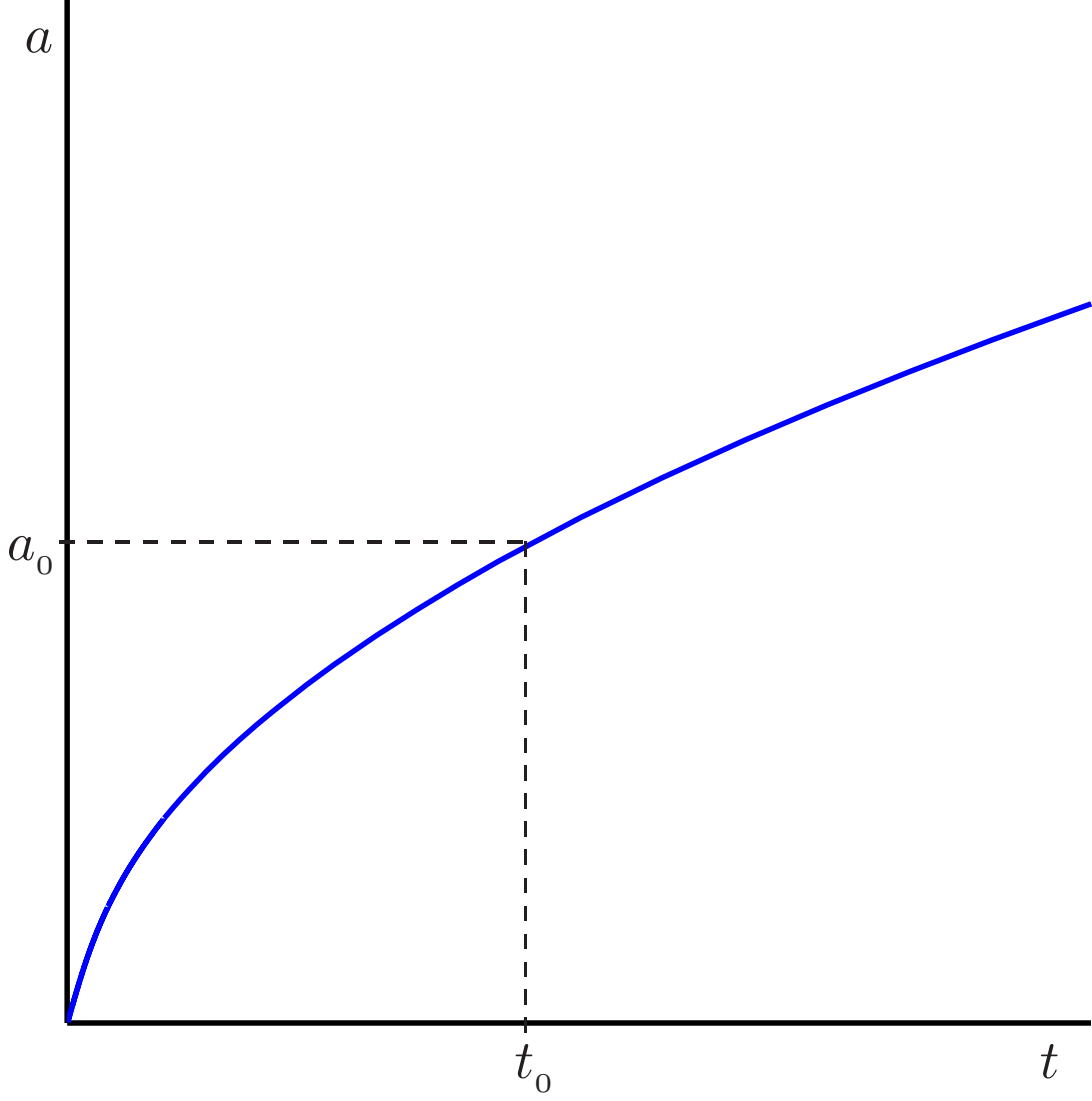


FIG. 18. Solution of  $\dot{a} = \sqrt{Aa^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right)}$  with  $A < 0$ ,  $k = 0$  and  $\dot{a}_0 < \dot{a}_{\max}$ .

whose approximate solution is

$$a(t) = a_0 \sqrt{1 \pm \sqrt{\frac{2\kappa_5 \rho_0}{3a_0^4}} (t - t_0)} \quad (157)$$

where we use  $B = \frac{\kappa_5 \rho_0}{3}$ .



### C. Case $k = 1$

In this case, the equation (99) can be rewritten as

$$\left(\frac{\dot{a}^2 + 1}{a^2}\right)^2 - 2A \left(\frac{\dot{a}^2 + 1}{a^2}\right) + AB \frac{a_0^4}{a^4} = 0, \quad (158)$$

from where

$$\dot{a} = \pm \sqrt{Aa^2 \left(1 \pm \text{sgn}(A) \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}}\right) - k} \quad (159)$$

with  $k = 1$ .

#### 1. Case $\alpha > 0$

In this case

$$A = \frac{1}{\alpha l^2} > 0. \quad (160)$$

From (159) we can see that  $\dot{a}$  is well defined if

$$a_{\min} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0, \quad (161)$$

so that

$$B \leq A \quad \text{i.e.,} \quad \rho_0 \leq \rho_{\max} = \frac{3}{\kappa_5 \alpha l^2}. \quad (162)$$

With these considerations we can analyze if the radicand is positive in (159)

$$Aa^2 \left(1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}}\right) - k. \quad (163)$$

#### a. Plus or minus sign?

Let us consider  $\dot{a} > 0$ , the analysis of the case  $\dot{a} < 0$  is very similar

$$\dot{a} = \sqrt{Aa^2 \left(1 \pm \sqrt{1 - \frac{a_{\min}^4}{a^4}}\right) - k}. \quad (164)$$

The function  $\dot{a}(a)$  is monotonically increasing (decreasing) if we consider the plus (minus) sign in front of the square root.

From (164) we can see that exist  $\dot{a}_{\text{cri}}$

$$\dot{a}_{\text{cri}} := \dot{a}(a_{\text{min}}) = \sqrt{\sqrt{\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0^2 - k}. \quad (165)$$

If we consider the plus(minus) sign in front of the square root,  $\dot{a}_{\text{cri}}$  is the minimum(maximum) value of  $\dot{a}$ .

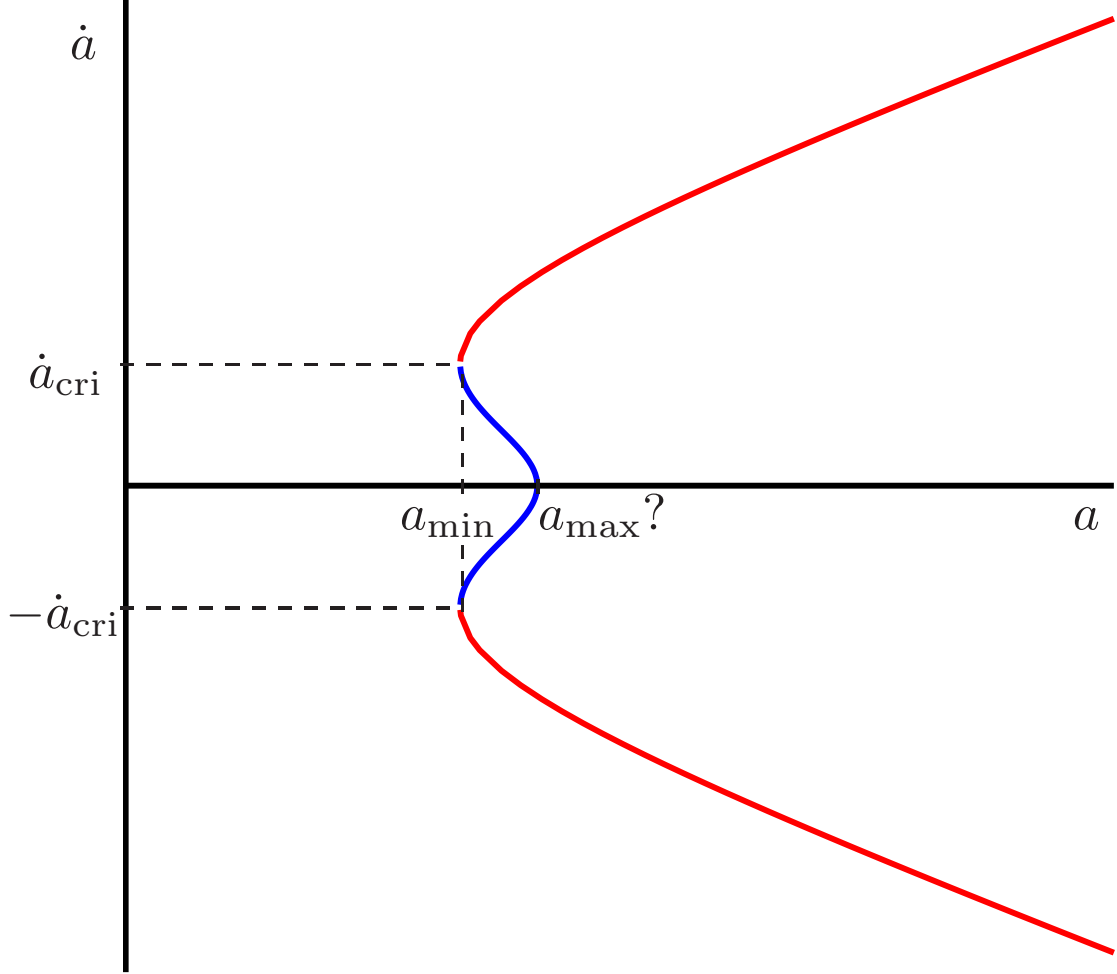


FIG. 19. Phase space for  $A > 0$  and  $k = 1$ .

*b. Case where the sign is “+”*

In this case

$$Aa^2 \left( 1 + \sqrt{1 - \frac{a_{\text{min}}^4}{a^4}} \right) - k \geq Aa_{\text{min}}^2 - k \geq 0, \quad (166)$$

so that

$$a_{\text{min}} \geq \sqrt{\frac{k}{A}} \iff \rho_0 a_0^4 \geq 3 \frac{\alpha l^2 k^2}{\kappa_5}, \quad (167)$$

but (see equation (98))

$$\rho(t) = \left( \frac{a_0}{a(t)} \right)^4 \rho_0 \implies \rho a^4 = \rho_0 a_0^4,$$

then

$$\rho a^4 \geq 3 \frac{\alpha l^2 k^2}{\kappa_5}. \quad (168)$$

It is direct to prove that  $\dot{a} \neq 0$  for  $a > a_{\min}$ , then  $a$  has no local maximums/minimums, and therefore  $a$  is monotonically increasing or monotonically decreasing.

If there is a limit to  $a \gg a_{\min}$ , then

$$\dot{a} = \pm \sqrt{Aa^2 \left( 1 + \sqrt{1 - \frac{a_{\min}^4}{a^4}} \right) - k} \approx \pm \sqrt{2Aa^2 - k} \quad (169)$$

whose approximate solution is

$$a(t) = \pm \sqrt{\frac{\alpha l^2 k}{2}} \times \cosh \left[ \sqrt{\frac{2}{\alpha l^2}} (t - t_0) + \operatorname{arcosh} \left( \sqrt{\frac{2}{\alpha l^2 k}} a_0 \right) \right]$$

where we use  $A = \frac{1}{\alpha l^2}$  and  $k = 1$ .

*c. Case where the sign is “-”*

In this case

$$Aa^2 \left( 1 - \sqrt{1 - \frac{a_{\min}^4}{a^4}} \right) - k \geq 0, \quad (170)$$

therefore

$$\frac{Aa^2 - k}{Aa^2} \geq \sqrt{1 - \frac{a_{\min}^4}{a^4}}. \quad (171)$$

This condition must be also satisfied by  $a_{\min}$

$$Aa_{\min}^2 - k \geq 0 \iff a_{\min} \geq \sqrt{\frac{k}{A}}, \quad (172)$$

so that,

$$\rho_0 a_0^4 \geq 3 \frac{\alpha l^2 k^2}{\kappa_5}, \quad (173)$$

but (see equation (98))

$$\rho a^4 = \rho_0 a_0^4$$

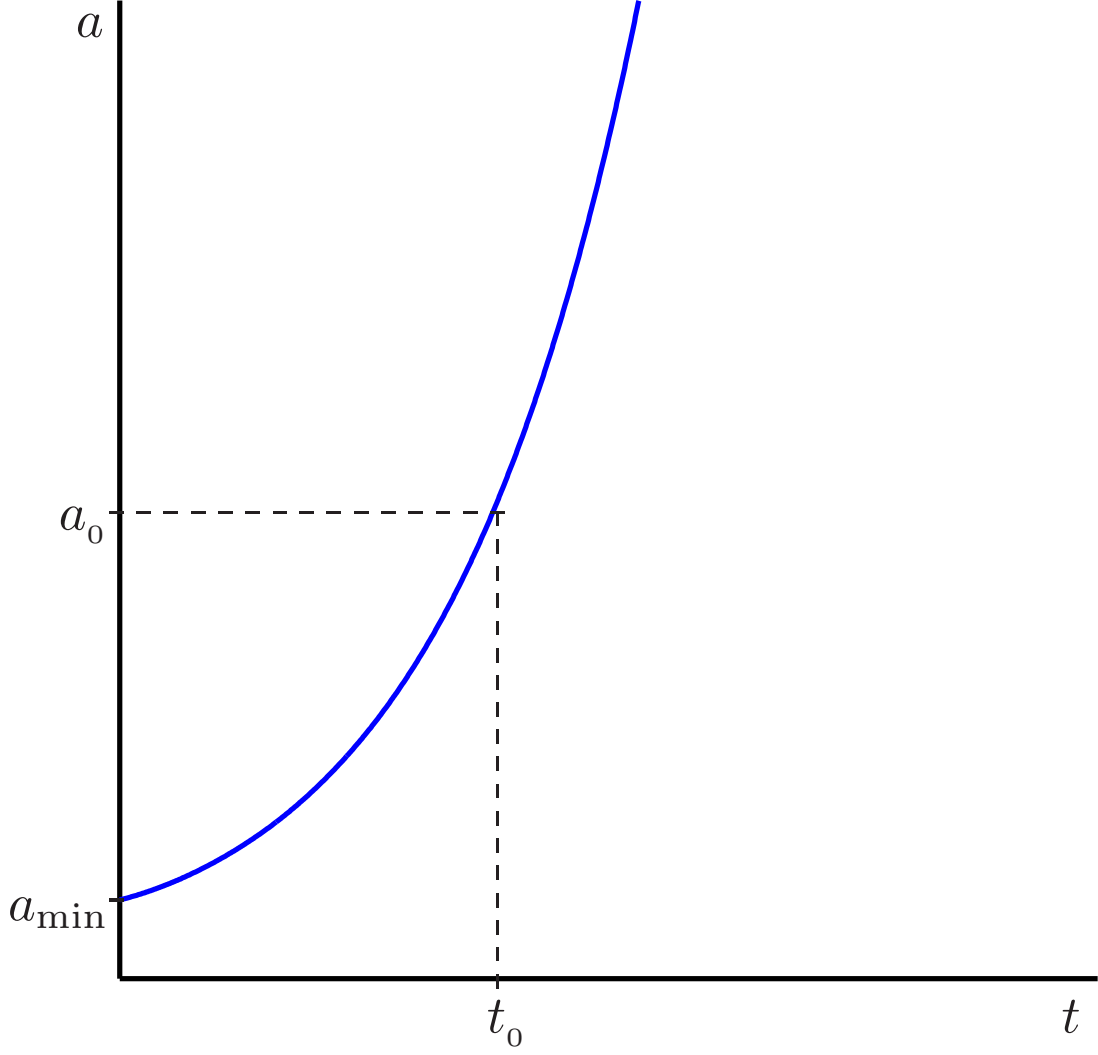


FIG. 20. Solution of  $\dot{a} = \sqrt{Aa^2 \left(1 - \sqrt{1 + \frac{a_{\min}^4}{a^4}}\right)} - k$  with  $A > 0$ ,  $k = 1$  and  $\dot{a}_0 > \dot{a}_{\text{cri}}$ .

and therefore

$$\rho a^4 \geq 3 \frac{\alpha l^2 k^2}{\kappa_5}. \quad (174)$$

From (171) we obtain

$$a \leq a_{\max} = \sqrt{\frac{k^2 + A^2 a_{\min}^4}{2Ak}}, \quad (175)$$

i.e.,

$$a_{\max} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}}. \quad (176)$$

From (176) we have

$$\rho = \frac{a_0^4}{a^4} \rho_0, \quad (177)$$

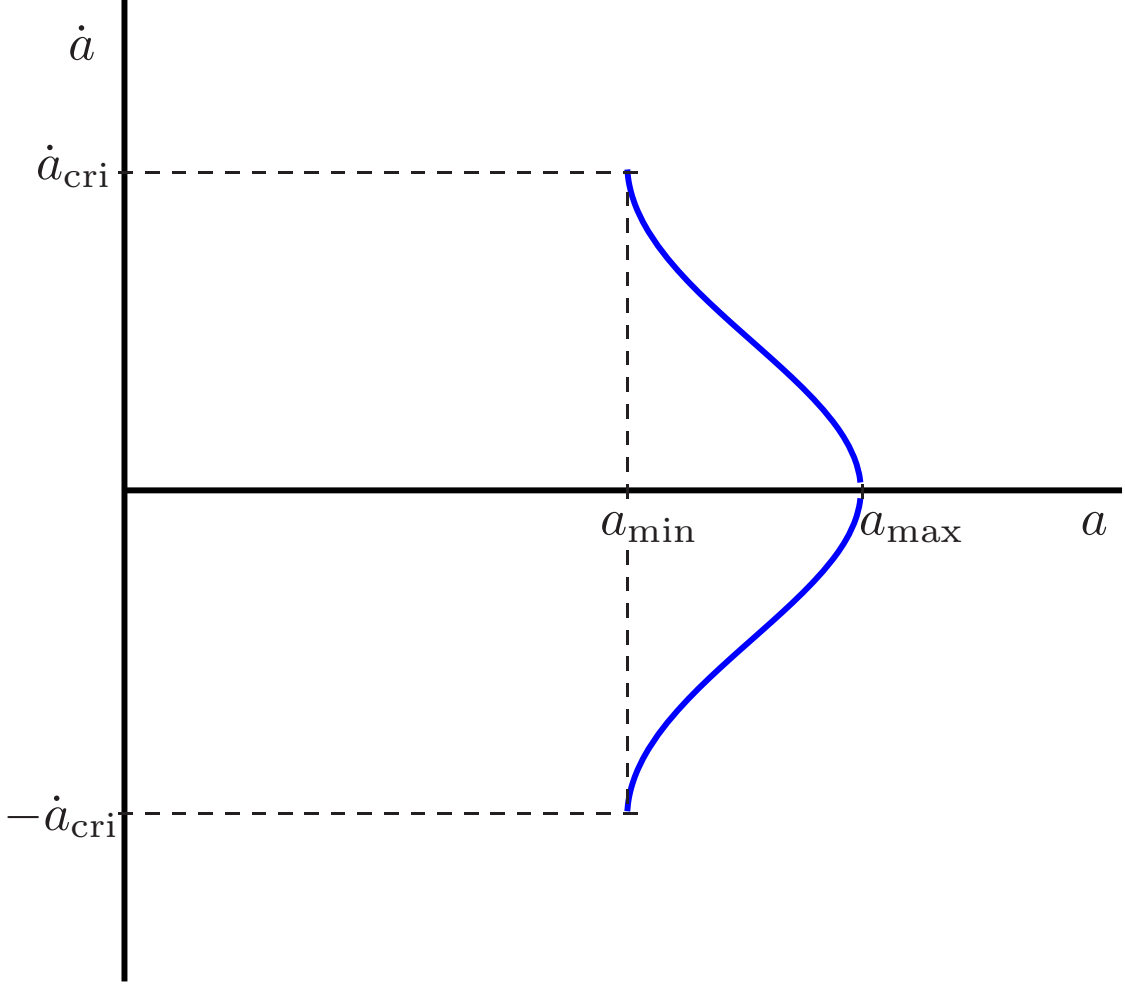


FIG. 21. Phase space for  $A > 0$  and  $k = 1$  with “ $-$ ” sign.

from where

$$\rho_{\min} = \frac{a_0^4}{a_{\max}^4} \rho_0 \quad (178)$$

and therefore

$$\rho_{\min} = \left( \frac{6ka_0^2}{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4} \right)^2 \rho_0 \quad (179)$$

.

## 2. Case $\alpha < 0$

In this case

$$A = \frac{1}{\alpha l^2} < 0. \quad (180)$$

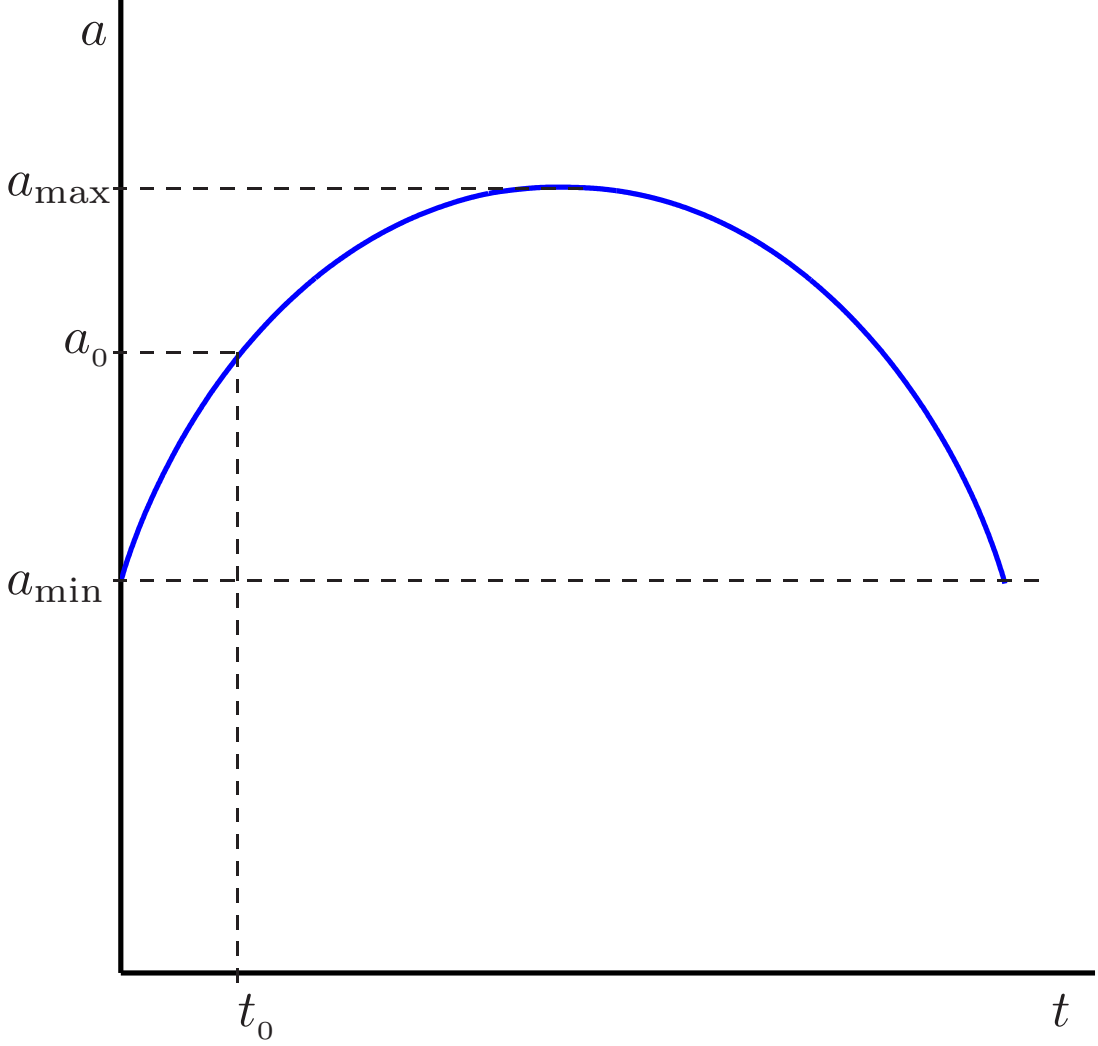


FIG. 22. Solution of  $\dot{a} = \sqrt{Aa^2 \left(1 - \sqrt{1 - \frac{a_{\min}^4}{a^4}}\right)} - k$  with  $A > 0$ ,  $k = 1$  and  $\dot{a}_0 < \dot{a}_{\text{cri}}$ .

From (159) we can see that  $\dot{a}$  is well defined if

$$Aa^2 \left(1 \pm \text{sgn}(A) \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}}\right) - k \geq 0. \quad (181)$$

this constrain exclude the case with plus sign “+” in front of square root. This condition leads

$$a \leq a_{\max} = \sqrt{\frac{-ABa_0^4 - k^2}{-2Ak}} \quad (182)$$

where  $k = 1$  and  $A = \frac{1}{\alpha l^2} < 0$ . There is a maximum value for  $a$

$$a_{\max} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}}, \quad (183)$$

this maximum leads

$$\rho_0 a_0^4 \geq -3 \frac{\alpha l^2 k^2}{\kappa_5}, \quad (184)$$

but (see equation (98))

$$\rho a^4 = \rho_0 a_0^4,$$

so that,

$$\rho a^4 \geq -3 \frac{\alpha l^2 k^2}{\kappa_5}. \quad (185)$$

If there is a maximum  $a_{\max}$  then, must exist a minimum for  $\rho$

$$\rho_{\min} = \left( \frac{6ka_0^2}{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4} \right)^2 \rho_0. \quad (186)$$

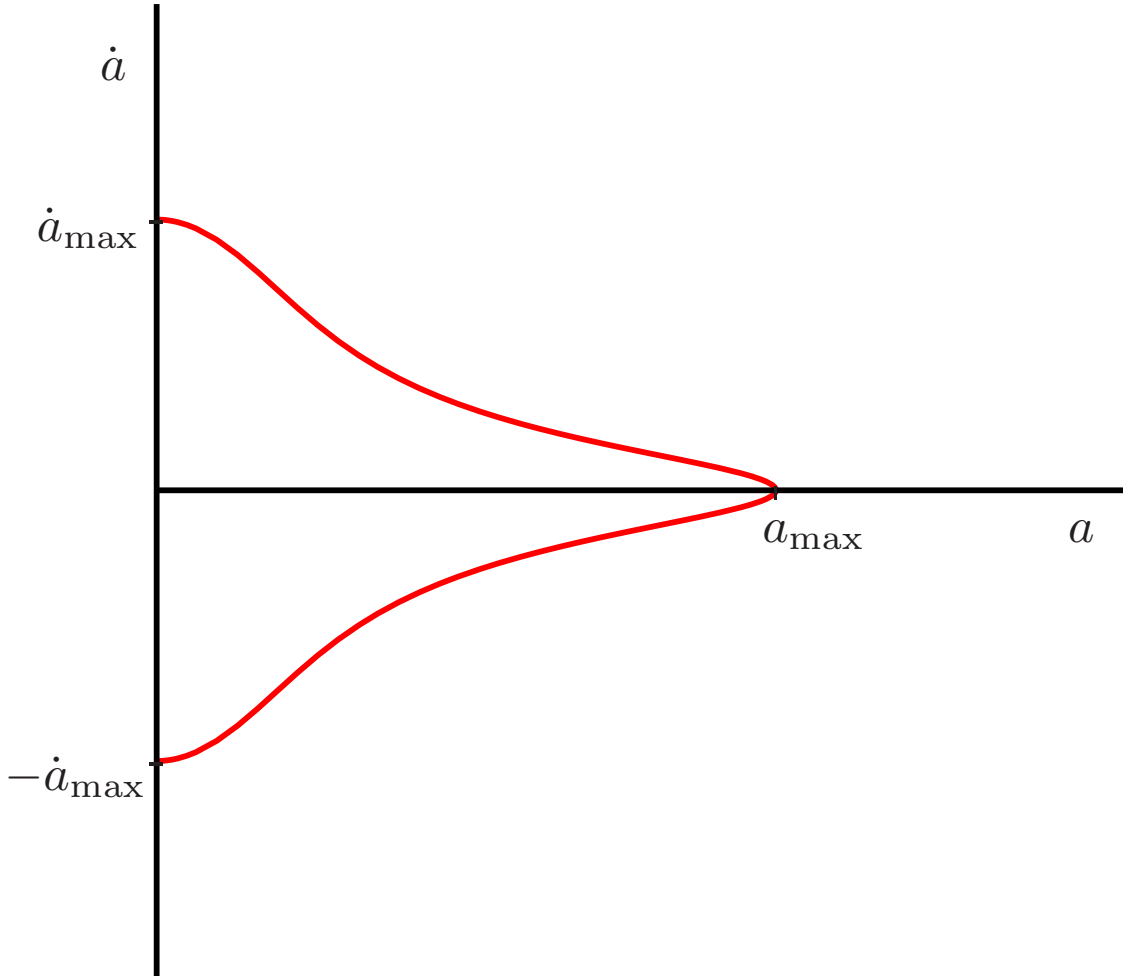


FIG. 23. Phase space for  $A < 0$  and  $k = 1$  with “ $-$ ” sign.

There is no a limit to  $a \rightarrow \infty$  and therefore it is impossible find an approximate solution for

$$\dot{a} = \pm \sqrt{Aa^2 \left( 1 - \sqrt{1 - \frac{B}{A} \frac{a_0^4}{a^4}} \right) - k}. \quad (187)$$

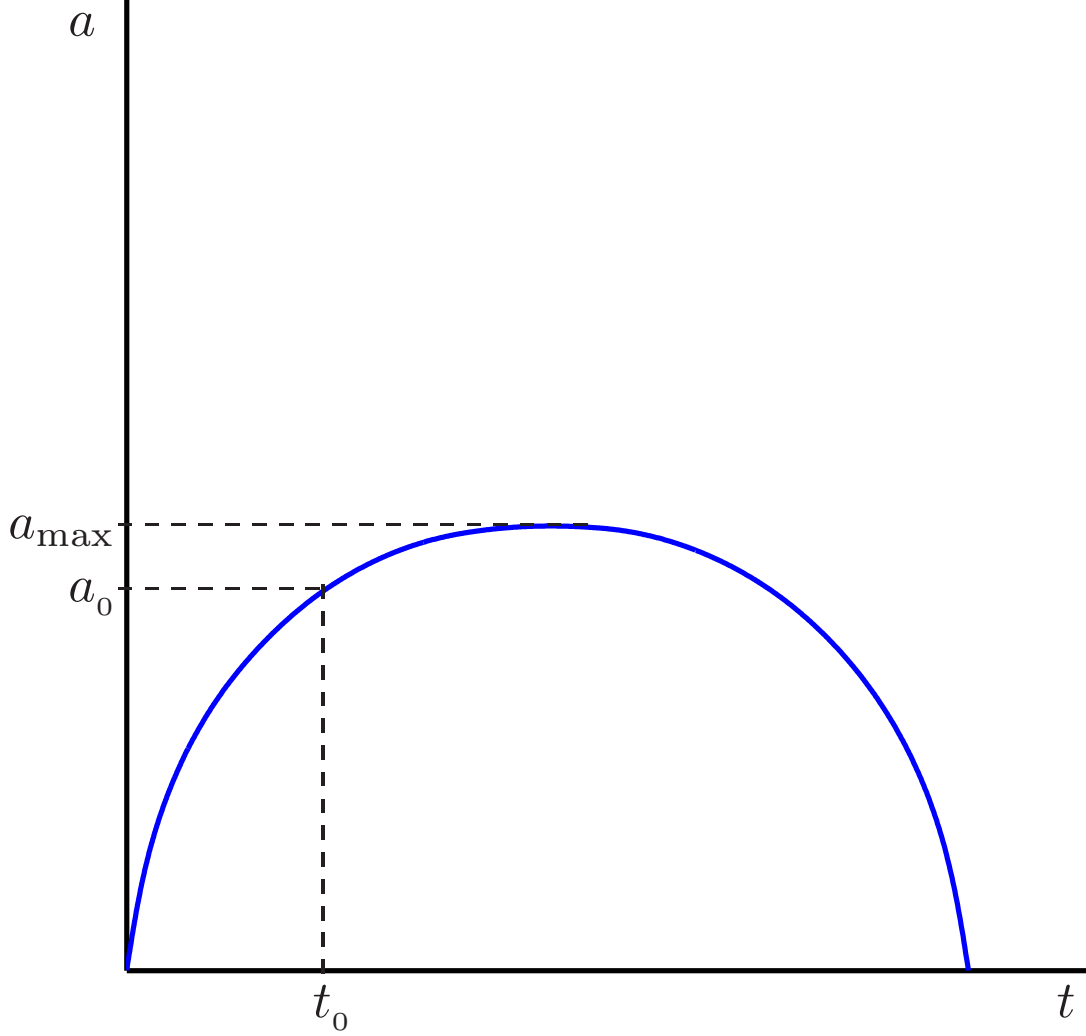


FIG. 24. Solution of  $\dot{a} = \sqrt{Aa^2 \left( 1 - \sqrt{1 - \frac{Ba_0^4}{Aa^4}} \right) - k}$  with  $A < 0$ ,  $k = 1$  and  $\dot{a}_0 < \dot{a}_{\max}$ .

#### D. Solutions for era of matter

We have found a family of solutions for era of matter.

If we consider an open space ( $k = -1$ ), the solutions found include (i) an accelerated expansion ( $\alpha > 0$ ) with a minimum scale factor at initial time that, when the time goes to infinity, the scale factor behaves as a hyperbolic sine function (Fig. 7) (ii) a decelerated



expansion ( $\alpha < 0$ ), with a Big Crunch in a finite time  $t_{\max}$  (Fig. 9) (iii) and a couple of solutions without accelerated expansion, whose scale factor tends to a constant value:  $\alpha > 0$  (Fig. 8) and  $\alpha < 0$  (Fig. 11). See Table IV and Table V.

TABLE IV. Expanding universe solutions for scale factor of an open space  $k = -1$  (hyperbolic) with  $\alpha > 0$ , where  $a_{\min} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0$ ,  $\omega = \sqrt{\frac{2}{\alpha l^2}}$ ,  $\phi = \text{arsinh}\left(\frac{\omega}{\sqrt{-k}} a_0\right)$ ,  $\rho_{\max} = \frac{3}{\kappa_5 \alpha l^2}$  and  $\dot{a}_{\text{cri}} = \sqrt{\sqrt{\frac{\kappa_5 \rho_0}{3 \alpha l^2}} a_0^2 - k}$ .

	Accelerated	No accelerated
$a$	$a_{\min} \leq a$	$a_{\min} \leq a$
$a(t \rightarrow \infty)$	$\sim \sinh\left(\omega(t - t_0) + \phi\right)$	$\sim (t - t_0)$
$\rho \sim \frac{1}{a^4}$	$0 < \rho \leq \rho_{\max}$	$0 < \rho \leq \rho_{\max}$
$\dot{a}$	$\dot{a}_{\text{cri}} < \dot{a}$	$\sqrt{-k} < \dot{a} < \dot{a}_{\text{cri}}$
$\dot{a}(t \rightarrow \infty)$	$\sim \cosh\left(\omega(t - t_0) + \phi\right)$	$\sim \sqrt{-k}$

TABLE V. Expanding universe solutions for scale factor of an open space  $k = -1$  (hyperbolic) with  $\alpha < 0$ , where  $a_{\max} = \sqrt{\frac{3 \alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}}$ ,  $\rho_{\min} = \left(\frac{6k a_0^2}{3 \alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}\right)^2 \rho_0$  and  $\dot{a}_{\max\pm} = \sqrt{\pm \sqrt{-\frac{\kappa_5 \rho_0}{3 \alpha l^2}} a_0^2 - k}$ . A decelerated solution describes *Big Crunch* in a finite time  $t_{\max}$ .

	Decelerated	No accelerated
$a$	$0 \leq a \leq a_{\max}$	$0 \leq a$
$a(t \rightarrow \infty)$	—	$\sim (t - t_0)$
$\rho \sim \frac{1}{a^4}$	$\rho_{\min} \leq \rho$	$0 < \rho$
$\dot{a}$	$-\dot{a}_{\max-} \leq \dot{a} \leq \dot{a}_{\max-}$	$\sqrt{-k} < \dot{a} \leq \dot{a}_{\max+}$
$\dot{a}(t \rightarrow \infty)$	—	$\sim \sqrt{-k}$

From models found in Section VIA we can see that there are solutions with  $\alpha > 0$  for accelerated contracting universe and no accelerated contracting universe (see Figure 6,  $\dot{a} < 0$ ). These solutions were not studied.

Solutions found for a flat universe ( $k = 0$ ) in expansion are (i) an accelerated expansion whose scale factor behaves as a exponential function when time grows and starts from a minimum value (Fig. 15) (ii) and a couple of solutions with decelerated expansion whose scale factor tends to square root function:  $\alpha > 0$  (Fig. 16) and  $\alpha < 0$  (Fig. 18). See Table

VI and Table VII.

TABLE VI. Expanding universe solutions for scale factor of a flat space  $k = 0$  with  $\alpha > 0$ , where

$$a_{\min} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0, \omega = \sqrt{\frac{2}{\alpha l^2}}, \rho_{\max} = \frac{3}{\kappa_5 \alpha l^2} \text{ and } \dot{a}_{\text{cri}} = \sqrt[4]{\frac{\kappa_5 \rho_0}{3 \alpha l^2}} a_0.$$

	Accelerated	Decelerated
$a$	$a_{\min} \leq a$	$a_{\min} \leq a$
$a(t \rightarrow \infty)$	$\sim \exp(\omega(t - t_0))$	$\sim \sqrt{1 + \omega(a_{\min}/a_0)^2 (t - t_0)}$
$\rho \sim \frac{1}{a^4}$	$0 < \rho \leq \rho_{\max}$	$0 < \rho \leq \rho_{\max}$
$\dot{a}$	$\dot{a}_{\text{cri}} < \dot{a}$	$0 < \dot{a} < \dot{a}_{\text{cri}}$
$\dot{a}(t \rightarrow \infty)$	$\sim \exp(\omega(t - t_0))$	$\sim \frac{1}{\sqrt{1 + \omega(a_{\min}/a_0)^2 (t - t_0)}}$

In this case there are also solutions of contraction universe ( $\dot{a} < 0$ ) (i) one ends with a minimum value  $a_{\min}$  when  $\alpha$  is positive (Fig. 14) (ii) and other ends with a Big Crunch when  $\alpha$  is negative (Fig. 17).

TABLE VII. Expanding universe solutions for scale factor of a flat space  $k = 0$  with  $\alpha < 0$ , where

$$a_{\text{ref}} = \sqrt[4]{-\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0, \omega = \sqrt{-\frac{2}{\alpha l^2}}, \text{ and } \dot{a}_{\max} = \sqrt[4]{-\frac{\kappa_5 \rho_0}{3 \alpha l^2}} a_0.$$

	Decelerate
$a$	$0 \leq a$
$a(t \rightarrow \infty)$	$\sim \sqrt{1 + \omega(a_{\text{ref}}/a_0)^2 (t - t_0)}$
$\rho \sim \frac{1}{a^4}$	$0 \leq \rho$
$\dot{a}$	$0 < \dot{a} \leq \dot{a}_{\max}$
$\dot{a}(t \rightarrow \infty)$	$\sim \frac{1}{\sqrt{1 + \omega(a_{\text{ref}}/a_0)^2 (t - t_0)}}$

Finally, we only found one solution for a closed universe ( $k = 1$ ) in expansion. This solution is found when  $\alpha$  is greater than zero. It behaves as a hyperbolic cosine function when time grows and starts from a minimum value (Fig 20). See Table VIII.

Furthermore, there are two contracting universe solutions, both ends in a finite time (i) one ends with a minimum value  $a_{\min}$ , when  $\alpha$  is positive (Fig. 22) (ii) and other ends with a Big Crunch, when  $\alpha$  is negative (See Table IX and Fig. 24).

TABLE VIII. Expanding universe solutions for scale factor of a closed space  $k = 1$  with  $\alpha > 0$ , where  $a_{\min} = \sqrt[4]{\frac{\kappa_5 \alpha l^2 \rho_0}{3}} a_0$ ,  $a_{\max} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}}$ ,  $\omega = \sqrt{\frac{2}{\alpha l^2}}$ ,  $\phi = \text{arcosh}\left(\frac{\omega}{\sqrt{k}} a_0\right)$ ,  $\rho_{\min} = \left(\frac{6ka_0^2}{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}\right)^2 \rho_0$ ,  $\rho_{\max} = \frac{3}{\kappa_5 \alpha l^2}$  and  $\dot{a}_{\text{cri}} = \sqrt{\sqrt{\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0^2 - k}$ . A decelerated solution describes an expanding universe, which then stops the expansion and then contracts until scale factor reaches a minimum  $a_{\min} > 0$ , in a finite time  $t_{\max}$ .

	Accelerated	Decelerated
$a$	$a_{\min} \leq a$	$a_{\min} \leq a \leq a_{\max}$
$a(t \rightarrow \infty)$	$\sim \cosh(\omega(t - t_0) + \phi)$	—
$\rho \sim \frac{1}{a^4}$	$0 < \rho \leq \rho_{\max}$	$\rho_{\min} \leq \rho \leq \rho_{\max}$
$\dot{a}$	$\dot{a}_{\text{cri}} < \dot{a}$	$-\dot{a}_{\text{cri}} < \dot{a} < \dot{a}_{\text{cri}}$
$\dot{a}(t \rightarrow \infty)$	$\sim \sinh(\omega(t - t_0) + \phi)$	—

TABLE IX. Expanding universe solutions with Big Crunch for scale factor of a closed space  $k = 1$  with  $\alpha < 0$ , where  $a_{\max} = \sqrt{\frac{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}{6k}}$ ,  $\rho_{\min} = \left(\frac{6ka_0^2}{3\alpha l^2 k^2 + \kappa_5 \rho_0 a_0^4}\right)^2 \rho_0$  and  $\dot{a}_{\max} = \sqrt{\sqrt{-\frac{\kappa_5 \rho_0}{3\alpha l^2}} a_0^2 - k}$ . This solution describes a expanding universe, which then stops the expansion and then contracts until a Big Crunch, in a finite time  $t_{\max}$ .

	Decelerate
$a$	$a \leq a_{\max}$
$a(t \rightarrow \infty)$	—
$\rho \sim \frac{1}{a^4}$	$\rho_{\min} \leq \rho$
$\dot{a}$	$-\dot{a}_{\max} < \dot{a} < \dot{a}_{\max}$
$\dot{a}(t \rightarrow \infty)$	—

## VII. SUMMARY

We have considered a five-dimensional Einstein-Chern-Simons action  $S = S_g + S_M$  which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Chern-Simons gravity action instead of the Einstein-Hilbert action and where the matter sector is given by the so called perfect fluid. We have shown that

- i The Einstein-Chern-Simons field equations (9 - 12) subject to the conditions  $T^a = 0$ ,

$k^{ab} = 0$  and  $\frac{\delta L_M}{\delta \omega^{ab}} = 0$  are re-written in a way similar to the Einstein Maxwell field equations (20 - 22). In the case where the equations (20 - 22) satisfy the cosmological principle and the ordinary matter is negligible compared to the dark energy, we find that the equations (20 - 22) take the form (41 - 45). When ordinary matter is modeled as dust (Era of Matter), we find that the equations (20 - 22) take the form (91 - 95).

- ii The field equations (41 - 45) were completely resolved for the age of Dark Energy (Sec. V, accelerated expansion). We find that the field  $h^a$  has a similar behavior to that of a cosmological constant.
- iii The field equations (91 - 95) were solved for the era of Matter (Sec. VI). We find several models that are consistent with standard cosmology. The dynamics of the field  $h^a$  (95) was not analyzed because the focus was placed on the dynamics of the scale factor  $a(t)$ .

In fact, in Section V we have found solutions that describes accelerated expansion for the three possible cosmological models of the universe. Namely, spherical expansion ( $k = 1$ ), flat expansion ( $k = 0$ ) and hyperbolic expansion ( $k = -1$ ) when the constant  $\alpha$  is greater than zero. This mean that the Einstein-Chern-Simons field equations have as a of their solutions an universe in accelerated expansion. This result allow us to conjeture that this solutions are compatible with the era of Dark Energy and that the energy-momentum tensor for the field  $h^a$  corresponds to a form of positive cosmological constant. We have also shown that the EChS field equations have solutions that allows us to identify the energy-momentum tensor for the field  $h^a$  with a negative cosmological constant.

On the other hand, in Section VI we have found a family of solutions for era of matter. In the case  $k = -1$  (open universe), the solutions correspond to (i) an accelerated expansion ( $\alpha > 0$ ) with a minimum scale factor at initial time that, when the time goes to infinity, the scale factor behaves as a hyperbolic sine function (ii) a decelerated expansion ( $\alpha < 0$ ), with a Big Crunch in a finite time  $t_{\max}$  (iii) and a couple of solutions without accelerated expansion, whose scale factor tends to a constant value. In the case  $k = 0$  (flat universe), the solutions describing (i) an accelerated expansion whose scale factor behaves as a exponential function when time grows and starts from a minimum value (ii) and a couple of solutions with decelerated expansion whose scale factor tends to square root function. In the case  $k = 1$  it is found only one solution for a closed universe in expansion, which behaves as a hyperbolic cosine function when time grows and starts from a minimum value. However

there are two contracting universe solutions, both ends in a finite time. One ends with a minimum value  $a_{\min}$ , when  $\alpha$  is positive and other ends with a Big Crunch, when  $\alpha$  is negative.

In summary, we have found some solutions for the field equations, which were obtained from a Lagrangian for a Chern-Simons gravity theory, studied in Ref. [1]. One problem with these solutions is that they are valid only in a five-dimensional space.

A connection between five-dimensional spacetimes and the four-dimensional universe could be accomplished by using a procedure, based on the Kaluza-Klein theory, known as dynamic compactification [12], [13]. The method consists in considering a spacetime metric in which the scale factor of the compact space evolves as an inverse power of the radius of the observable universe. In fact the metric can be written in a convenient way so that it can achieve the compactness of the fifth dimension. Following refs. [12], [13] we could consider the 5-dimensional metric

$$ds^2 = -dt^2 + a^2(t) \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + b^2(t) dx^4, \quad (188)$$

and then consider the case when the scale factor  $b(t)$  is given by

$$b(t) = \frac{1}{a^n}, \quad n > 0. \quad (189)$$

where the parameter  $n$  must be positive for dynamical compactification to take place.

Substituting (189) into the metric (188) we have

$$ds^2 = -dt^2 + a^2(t) \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + \frac{dx^4^2}{a^{2n}(t)}. \quad (190)$$

Therefore  $b$  gets smaller as the radius of our universe  $a$  becomes bigger.

It is possible to conjecture that the dynamic compactification procedure could lead, in a certain limit, to the usual results of the 4-dimensional general relativity (work in progress).

It should be noted that this compactification procedure, (dynamic compactification) can not be directly implemented on the theory, because this gravity theory is a theory based on a Chern-Simons Lagrangian.

In Ref. [14], subsequently Ref. [15], [16], [17] and most recently Ref.[18] was pointed out that Chern-Simons theories are connected with some even-dimensional structures known as gauged Wess-Zumino-Witten ( $gWZW$ ) terms. In Refs. [19], [20], [21], was shown that a five-dimensional Chern-Simons action invariant under the generalized Poincare algebra

$\mathfrak{B}_5$  induces a gauged Wess-Zumino-Witten term containing the four-dimensional Einstein-Hilbert action.

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## Appendix A: Obtaining equations (35-39)

From equations (28-32) of Ref. [8] we know that

$$48\alpha_3 \left( \frac{\dot{a}^2 + k}{a^2} \right) + 24\alpha_1 l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \beta_1 T_{00}, \quad (\text{A1})$$

$$-24\alpha_3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] - 24\alpha_1 l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = \beta_1 T_{11}, \quad (\text{A2})$$

$$24\alpha_3 l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \beta_2 T_{00}^{(h)}, \quad (\text{A3})$$

$$-24\alpha_3 l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = \beta_2 T_{11}^{(h)}, \quad (\text{A4})$$

$$24\alpha_3 l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (g - f) \frac{\dot{a}}{a} + \dot{g} \right] = 0 \quad (\text{A5})$$

where

$$h^0 = f(t) e^0 \quad (\text{A6})$$

$$h^p = g(t) e^p, \quad p = 1, \dots, 4. \quad (\text{A7})$$

In this article we have considered  $\beta_1 = \beta_2 = \kappa$ . Making this replacement in (A1-A7) and dividing it by  $8\alpha_3$  we have

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) + \left( \frac{\alpha_1}{\alpha_3} \right) \left[ 3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 \right] = \left( \frac{\kappa}{8\alpha_3} \right) T_{00}, \quad (\text{A8})$$

$$\begin{aligned} -8 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] \\ - \left( \frac{\alpha_1}{\alpha_3} \right) \left[ 3l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = \left( \frac{\kappa}{8\alpha_3} \right) T_{11}, \end{aligned} \quad (\text{A9})$$

$$3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \left( \frac{\kappa}{8\alpha_3} \right) T_{00}^{(h)}, \quad (\text{A10})$$

$$-3l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = \left( \frac{\kappa}{8\alpha_3} \right) T_{11}^{(h)}, \quad (\text{A11})$$

$$3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (g - f) \frac{\dot{a}}{a} + \dot{g} \right] = 0. \quad (\text{A12})$$

Consider now the definition of the constants of Section II

$$\kappa_5 = \frac{\kappa}{8\alpha_3}, \quad \alpha = -\frac{\alpha_1}{\alpha_3}. \quad (\text{A13})$$

With these constants, equations (A8 - A12) take the form

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) - \alpha \left[ 3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 \right] = \kappa_5 T_{00}, \quad (\text{A14})$$

$$-8 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] + \alpha \left[ 3l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = \kappa_5 T_{11}, \quad (\text{A15})$$

$$3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \kappa_5 T_{00}^{(h)}, \quad (\text{A16})$$

$$-3l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 T_{11}^{(h)}, \quad (\text{A17})$$

$$3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (g - f) \frac{\dot{a}}{a} + \dot{g} \right] = 0. \quad (\text{A18})$$

Replacing now (A16) in square brackets (A14), and (A17) in square brackets (A15), and passing those terms on the right side of the equations, we find

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 T_{00} + \kappa_5 \alpha T_{00}^{(h)}, \quad (\text{A19})$$

$$-8 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = \kappa_5 T_{11} + \alpha \kappa_5 T_{11}^{(h)}, \quad (\text{A20})$$

$$3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \kappa_5 T_{00}^{(h)}, \quad (\text{A21})$$

$$-3l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 T_{11}^{(h)}, \quad (\text{A22})$$

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (g - f) \frac{\dot{a}}{a} + \dot{g} \right] = 0. \quad (\text{A23})$$

Accommodating some signs in Eqs. (A20) and (A22), grouping some terms (Eqs. A19 and A20), we have

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 \left( T_{00} + \alpha T_{00}^{(h)} \right), \quad (\text{A24})$$

$$8 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \left( T_{11} + \alpha T_{11}^{(h)} \right), \quad (\text{A25})$$

$$3l^2 \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \kappa_5 T_{00}^{(h)}, \quad (\text{A26})$$

$$3l^2 \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -\kappa_5 T_{11}^{(h)}, \quad (\text{A27})$$

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (g - f) \frac{\dot{a}}{a} + \dot{g} \right] = 0. \quad (\text{A28})$$

In Section IV was considered an energy-momentum tensor of the form

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \alpha T_{\mu\nu}^{(h)} \quad (\text{A29})$$

$$= \text{diag}(\rho, p, p, p, p) + \alpha \text{diag}(\rho^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}) \quad (\text{A30})$$

$$= \text{diag}(\rho + \alpha \rho^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}, p + \alpha p^{(h)}) \quad (\text{A31})$$

$$= \text{diag}(\tilde{\rho}, \tilde{p}, \tilde{p}, \tilde{p}, \tilde{p}) \quad (\text{A32})$$



where

$$T_{\mu\nu} = \text{diag}(\rho, p, p, p, p) \quad (\text{A33})$$

$$T_{\mu\nu}^{(h)} = \text{diag}(\rho^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}, p^{(h)}) . \quad (\text{A34})$$

Writing the functions  $f$  and  $g$  as

$$f = h(0), \quad g = h, \quad (\text{A35})$$

we find that the equations (A24 - A28) take the form

$$6 \left( \frac{\dot{a}^2 + k}{a^2} \right) = \kappa_5 \tilde{\rho}, \quad (\text{A36})$$

$$3 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}^2 + k}{a^2} \right) \right] = -\kappa_5 \tilde{p}, \quad (\text{A37})$$

$$\frac{3l^2}{\kappa_5} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 = \rho^{(h)}, \quad (\text{A38})$$

$$\frac{3l^2}{\kappa_5} \frac{\ddot{a}}{a} \left( \frac{\dot{a}^2 + k}{a^2} \right) = -p^{(h)}, \quad (\text{A39})$$

$$\left( \frac{\dot{a}^2 + k}{a^2} \right) \left[ (h - h(0)) \frac{\dot{a}}{a} + \dot{\varphi} \right] = 0. \quad (\text{A40})$$

which correspond to the equations (35 - 39).

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